AN INTRODUCTION

TO THE

MATHEMATICAL

THEORY OF ATTRACTION.

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PREFACE.

My time for some years was so much occupied by administrative duties in Trinity College that I was unable to make any attempt to carry out the intentions stated above fourteen years ago in the Preface to the first volume of this treatise.

I have now to some extent accomplished what I then proposed. I came, however, to the conclusion that a chapter on Conjugate Functions was not suited for such a treatise as the present, and that to a student having a limited amount of time at his disposal some account of Maxwell's Theory of Light would be more interesting and instructive. This theory is not of course part of the Theory of Attraction, but is so intimately connected with the properties of magnetized bodies, electric currents, and dielectrics treated of in the present volume that its introduction does not seem unsuitable.

I should recommend a student reading this book for the first time to omit the whole of Chapter VIII after Article 146.

Of the more recent developments of the electromagnetic theory of light I have not attempted to give any account. So far as I can judge some of these rest on insecure foundations. I imagine, however, that before studying the most recent investigations a preliminary knowledge of Maxwell's theory is required, and I trust, therefore, that my chapter on the subject will not be entirely useless to the student.

I have to thank Mr. S. B. Kelleher, F.T.C.D., for his kindness in reading the proof-sheets of this book, and furnishing me with many valuable corrections.

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MATHEMATICAL THEORY OF ATTRACTION.

CHAPTER VIII.

SPHERICAL AND ELLIPSOIDAL HARMONICS.

SECTION I.—*Spherical Surfaces.*

137. **Expansion of Potential in Series of Solid Harmonics.**—It was shown in Art. 78 that the potential V at a point P, more distant than any point in the attracting mass from the origin, can be expanded in a series of descending powers of r, where r denotes the distance of P from the origin.

In this case, the series for the potential is of the form

$$\frac{M}{r} + \Sigma \frac{Y_n}{r^{n+1}},$$

where M denotes the attracting mass, and Y_1, Y_2, &c., are functions of θ and ϕ, the angular coordinates of P, and of constants depending on the attracting mass, but independent of the position of P.

Since $\nabla^2 V = 0$ for all positions of P outside M, the coefficient of each power of r in $\nabla^2 V$ must vanish separately, and therefore

$$\nabla^2 \left(\frac{Y_n}{r^{n+1}}\right) = 0.$$

Using for ∇^2 the expression given, equation (17), Art. 48, we obtain

$$\frac{d}{d\mu}(1-\mu^2)\frac{dY_n}{d\mu} + \frac{1}{1-\mu^2}\frac{d^2 Y_n}{d\phi^2} + n(n+1) Y_n = 0. \quad (1)$$

If Y be a function of θ, and ϕ satisfying (1), it is easily seen that $\nabla^2(r^n Y) = 0$; accordingly, if $V_n = r^n Y_n$, we have $\nabla^2 V_n = 0$, and V_n *is a homogeneous function of x, y, z of the degree n satisfying Laplace's equation.* Such a function is called *a spherical solid harmonic of the degree n.*

It appears from what has been said that if V_n denote a solid harmonic of the degree n, then $r^{-(2n+1)} V_n$ is also a solid harmonic whose degree is $-(n+1)$.

The function $\dfrac{V_n}{r^n}$ is termed a *spherical surface harmonic* of the degree n, and is what has been denoted above by Y_n.

In the present case, by considering the expression from whose expansion $\dfrac{V_n}{r^{2n+1}}$ was obtained, it is easy to see that V_n is a rational and integral function of x, y, z. In what follows, V_n will be termed a solid, and Y_n a spherical harmonic.

138. **Laplace's and Legendre's Coefficients.**—If the attracting mass be concentrated at a point Q whose polar coordinates are r', θ', ϕ', and whose distance from any point P is r, we have

$$V = \frac{M}{\mathrm{r}} = \frac{M}{r}\left(1 - 2\lambda\frac{r'}{r} + \frac{r'^2}{r^2}\right)^{-\frac{1}{2}} = \frac{M}{r'}\left(1 - 2\lambda\frac{r}{r'} + \frac{r^2}{r'^2}\right)^{-\frac{1}{2}},$$

where $\qquad \lambda = \mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\phi - \phi')$.

In this case, if P be farther than Q from the origin,

$$V = M\Sigma L_n \frac{r'^n}{r^{n+1}};$$

and if P be nearer the origin,

$$V = M\Sigma L_n \frac{r^n}{r'^{n+1}}.$$

The coefficients L_1, L_2, &c., in the development of r^{-1} are called *Laplace's Coefficients.* They are obviously spherical harmonics of a particular kind. They may be defined as the *coefficients of the successive powers of h in the expansion of*

$$(1 - 2\lambda h + h^2)^{-\frac{1}{2}}.$$

These coefficients are plainly symmetrical with respect to the angular coordinates of P and Q.

If the point Q be on the axis from which θ is counted, $\lambda = \mu$, and Laplace's coefficients become *the coefficients of the successive powers of h in the development of*

$$(1 - 2\mu h + h^2)^{-\frac{1}{2}}.$$

In this case, these coefficients are functions of μ solely, and are called *Legendre's Coefficients.* They are usually denoted by P_1, P_2, &c.

It is plain that P_n satisfies the equation

$$\frac{d}{d\mu}(1 - \mu^2)\frac{dP_n}{d\mu} + n(n+1)P_n = 0. \qquad (2)$$

In general, a spherical harmonic of the degree n, which is a function of μ solely, satisfies (2), and is called a *zonal harmonic.*

139. **Properties of Complete Spherical Harmonics.**—A spherical harmonic which when expressed as a function of the coordinates is *finite and single-valued* for all points of space, is said to be *complete.* If Y_m and Y_n be complete spherical harmonics of different degrees,

$$\int_{-1}^{+1}\int_0^{2\pi} Y_m Y_n \, d\mu \, d\phi = 0. \qquad (3)$$

This may be proved as follows:—It appears from Art. 137 that

$$\frac{Y_m}{r^{m+1}} \quad \text{and} \quad \frac{Y_n}{r^{n+1}}$$

satisfy Laplace's equation; and by (5), Art. 58, if we take as the field of integration the space outside a sphere S of radius a described round the origin as centre, we have

$$\int \frac{Y_m}{r^{m+1}} \frac{d}{dr}\left(\frac{Y_n}{r^{n+1}}\right) dS = \int \frac{Y_n}{r^{n+1}} \frac{d}{dr}\left(\frac{Y_m}{r^{m+1}}\right) dS.$$

Hence

$$\frac{n+1}{a^{m+n+1}} \iint Y_m Y_n \, d\mu \, d\phi = \frac{m+1}{a^{m+n+1}} \iint Y_m Y_n \, d\mu \, d\phi,$$

and therefore, unless $m = n$, equation (3) must hold good.

If Y_n be a complete spherical harmonic of the degree n, and L_n a Laplacian coefficient of the same degree,

$$\int_{-1}^{+1}\int_0^{2\pi} L_n Y_n \, d\mu \, d\phi = \frac{4\pi}{2n+1} Y'_n. \qquad (4)$$

To prove this, take as the field of integration the space outside a sphere S whose centre is at the origin, and whose radius a is less than r', the distance of the point Q from the origin, and let r denote the distance of any point from Q; then the function $\frac{Y_n}{r^{n+1}}$ satisfies Laplace's equation, and therefore by (10), Art. 59, we have

$$\int \frac{Y_n}{a^{n+1}} \frac{d}{dr}\left(\frac{1}{\mathrm{r}}\right) dS - 4\pi \frac{Y'_n}{r'^{n+1}} = -\int \frac{n+1}{\mathrm{r}} \frac{Y_n}{a^{n+2}} \, dS;$$

but

$$\frac{1}{\mathrm{r}} = \Sigma L_n \frac{r^n}{r'^{n+1}}$$

at all points for which $r < r'$, whence at the surface S we have

$$\frac{d}{dr}\left(\frac{1}{\mathrm{r}}\right) = \Sigma n L_n \frac{\mathrm{r}^{n-1}}{r'^{n+1}};$$

also $dS = a^2 d\mu \, d\phi$, and

$$\iint L_m Y_n \, d\mu \, d\phi = 0,$$

unless $m = n$. Hence we obtain

$$\frac{n}{r'^{n+1}} \iint L_n Y_n \, d\mu \, d\phi - 4\pi \frac{Y'_n}{r'^{n+1}} = -\frac{n+1}{r'^{n+1}} \iint L_n Y_n \, d\mu \, d\phi,$$

from which equation (4) follows by transposition.

If two series of spherical harmonics are equal for all values of μ and ϕ, each harmonic of one series is equal to the harmonic of the other series whose degree is the same.

Here

$$Y_0 + Y_1 + Y_2 + \&c. = Z_0 + Z_1 + Z_2 + \&c.$$

If each side of this equation be multiplied by L_n and integrated, since

$$\iint L_n Y_m \, d\mu \, d\phi = 0,$$

by (4) we obtain

$$\frac{4\pi}{2n+1} Y'_n = \frac{4\pi}{2n+1} Z'_n;$$

and as this equation holds good for all values of μ' and ϕ', we get $Y_n = Z_n$.

Any function of μ and ϕ which is finite and single-valued can be expanded in a series of spherical harmonics.

The method of arriving at this result is suggested by what has been already proved. If it be possible to express $f(\mu\phi)$ in the form ΣY_n, we must have

$$\begin{aligned} 4\pi f(\mu'\phi') = 4\pi \Sigma Y'_n &= \Sigma(2n+1) \int_{-1}^{+1} \int_0^{2\pi} L_n Y_n \, d\mu \, d\phi \\ &= \int_{-1}^{+1} \int_0^{2\pi} f(\mu\phi) \, \Sigma(2n+1) L_n \, d\mu \, d\phi. \qquad (5) \end{aligned}$$

Now $\qquad (1 - 2\lambda h + h^2)^{-\frac{1}{2}} = \Sigma L_n h^n;$

whence, differentiating and multiplying by $2h$, we have

$$\frac{2(\lambda h - h^2)}{(1 - 2\lambda h + h^2)^{\frac{3}{2}}} = \Sigma 2n L_n h^n,$$

then by addition to the former equation we get

$$\frac{1 - h^2}{(1 - 2\lambda h + h^2)^{\frac{3}{2}}} = \Sigma(2n+1) L_n h^n.$$

Accordingly, if the supposed expansion be possible, we must, when $h = 1$, have

$$4\pi f(\mu'\phi') = \int_{-1}^{+1} \int_0^{2\pi} f(\mu\phi) \frac{1 - h^2}{(1 - 2\lambda h + h^2)^{\frac{3}{2}}} \, d\mu \, d\phi; \qquad (6)$$

and conversely, if this equation be true, the expansion is possible.

That equation (6) is true can be shown in the following manner:—

Let Q be a point outside a sphere S whose centre is the origin and whose radius is a, and let r denote the distance of any point on S from Q. Then

$$\mathrm{r}^2 = a^2 + r'^2 - 2\lambda a r',$$

where r' denotes the distance of Q from the origin; and if $h = \dfrac{a}{r'}$, we have

$$\frac{1 - h^2}{(1 - 2\lambda h + h^2)^{\frac{3}{2}}} = \frac{r'(r'^2 - a^2)}{\mathrm{r}^3}.$$

As in Art. 42, we have

$$dS = \frac{2\pi a}{r'}\,\mathrm{r}\,d\mathrm{r},$$

and therefore

$$\int \frac{dS}{\mathrm{r}^3} = \frac{2\pi a}{r'}\left(\frac{1}{r' - a} - \frac{1}{r' + a}\right) = \frac{4\pi a^2}{r'(r'^2 - a^2)};$$

also $dS = a^2 d\mu\, d\phi$, and accordingly

$$\int_{-1}^{+1}\int_{0}^{2\pi} \frac{1 - h^2}{(1 - 2\lambda h + h^2)^{\frac{3}{2}}}\, d\mu\, d\phi = 4\pi.$$

The value of the definite integral above is therefore independent of r'; but $h = 1$ when $r' = a$, and in this case each element of the integral in (6) is zero, unless r be infinitely small, in which case $\mu = \mu'$, and $\phi = \phi'$. Hence, when $h = 1$, we have

$$\iint f(\mu\phi)\frac{1 - h^2}{(1 - 2\lambda h + h^2)^{\frac{3}{2}}}\, d\mu\, d\phi = f(\mu'\phi')\iint \frac{1 - h^2}{(1 - 2\lambda h + h^2)^{\frac{3}{2}}}\, d\mu\, d\phi$$

$$= 4\pi f(\mu'\phi').$$

140. **Application of Spherical Harmonics.**—When the potential is due to mass on one side of a spherical surface S and is given at each point of the surface S itself, the potential at any point on the side of S remote from the mass can be represented by a series of solid harmonics. At the surface S this series becomes a series of spherical harmonics representing the known value of the potential at the surface. Hence by Art. 139 each harmonic in this series is determined, and consequently so also are the corresponding solid harmonics representing the potential on one side of S.

If the potential be due to a distribution of mass on the surface S whose density is given, the potential outside S can be represented by the series

$$\frac{M}{r} + \Sigma \frac{a^n Y_n}{r^{n+1}},$$

and at any point inside by the series

$$\frac{M}{a} + \Sigma \frac{r^n Z_n}{a^{n+1}}$$

At all points of the surface these two expressions must be equal, whence by Art. 139, $Z_n = Y_n$. Again, if V and V' denote the potentials outside and inside the surface, we have by Art. 46 at the surface

$$\frac{dV}{dr} - \frac{dV'}{dr} + 4\pi\sigma = 0,$$

that is,

$$\frac{M}{a^2} + \Sigma (2n+1) \frac{Y_n}{a^2} = 4\pi\sigma = 4\pi \Sigma S_n ;$$

whence

$$M = 4\pi a^2 S_0, \quad Y_n = \frac{4\pi a^2}{2n+1} S_n,$$

and

$$V = 4\pi a^2 \Sigma \frac{a^n S_n}{r^{n+1}}, \quad V' = 4\pi a^2 \Sigma \frac{r^n S_n}{a^{n+1}}.$$

141. **Potential of Homogeneous Spheroid.**—If the surface of a solid differs but little from a sphere whose centre is at the origin, the radius vector r is given by an equation of the form $r = a(1 + \alpha y)$, where a denotes the radius of the sphere, y a function of the angular coordinates μ and ϕ, and α a small constant whose square may be neglected.

The potential V at any external point is the sum of the potential due to the sphere and of that due to the shell whose thickness at any point is $a\alpha y$. Hence if ρ denote the density of the spheroid, μ', ϕ' the coordinates of a point on the surface of the sphere, and r the distance of this point from the point r, μ, ϕ in external space, we have

$$V = \frac{4}{3}\frac{\pi\rho a^3}{r} + \alpha\int\frac{\rho a^3 y' \, d\mu' \, d\phi'}{\mathrm{r}};$$

but by Art. 139, $y = \Sigma Y_n$, and therefore by Art. 138, and by (3) and (4), we get

$$\left.\begin{aligned} V &= \frac{4}{3}\frac{\pi\rho a^3}{r} + \alpha\rho a^3\int\frac{L_n Y'_n a^n \, d\mu' \, d\phi'}{r^{n+1}} \\ &= \frac{4}{3}\frac{\pi\rho a^3}{r} + 4\alpha\pi\rho a^3 \sum \frac{a^n Y_n}{(2n+1)\, r^{n+1}} \end{aligned}\right\}. \qquad (7)$$

For the potential at a point inside the sphere, by Art. 42, we obtain, in like manner,

$$V = 2\pi\rho a^2 - \tfrac{2}{3}\pi\rho r^2 + 4\alpha\pi\rho a^3 \sum \frac{r^n Y_n}{(2n+1)\, a^{n+1}}. \qquad (8)$$

142. **Potential of Heterogeneous Spheroid.**—If a spheroid be composed of homogeneous layers comprised between surfaces given by equations of the form

$$r = a(1 + \alpha \Sigma Y_n),$$

where Y_n is a spherical harmonic which varies with the surface, and a is a variable parameter, we have, for the

potential δV of a single layer at a point outside, the equation

$$\delta V = 4\pi \frac{\rho a^2 da}{r} + 4a\pi\rho \sum \frac{d(a^{n+3} Y_n)}{(2n+1) r^{n+1}},$$

and at a point inside,

$$\delta V = 4\pi\rho a \, da + 4a\pi\rho \sum \frac{d(a^{2-n} Y_n)}{(2n+1)} r^n.$$

Hence for the potential V of a heterogeneous spheroid at a point outside it, if a_1 denote the parameter of the external surface, we obtain the equation

$$V = \frac{4\pi \int_0^{a_1} \rho a^2 da}{r} + 4a\pi \sum \frac{\int_0^{a_1} \rho d(a^{n+3} Y_n)}{(2n+1) r^{n+1}}. \tag{9}$$

For the potential of a heterogeneous shell comprised between surfaces whose parameters are a_1 and a_2, at an internal point, we get

$$V = 4\pi \int_{a_2}^{a_1} \rho a \, da + 4a\pi \sum \frac{\int_{a_2}^{a_1} \rho d(a^{2-n} Y_n)}{(2n+1)} r^n. \tag{10}$$

By combining the expressions given by (9) and (10), we find for the potential of a heterogeneous spheroid, at an internal point lying on a surface whose parameter is a, the equation

$$V = \frac{4\pi}{a} (1 - a\Sigma Y_n) \int_0^a \rho a^2 da + 4a\pi \sum \frac{\int_0^a \rho d(a^{n+3} Y_n)}{(2n+1) a^{n+1}}$$

$$+ 4\pi \int_a^{a_1} \rho a \, da + 4a\pi \sum \frac{\int_a^{a_1} \rho d(a^{2-n} Y_n)}{(2n+1)} a^n. \tag{11}$$

143. Homogeneous Mass of Revolving Fluid.— If a homogeneous mass of fluid revolving with a uniform angular velocity be in a state of relative equilibrium under its own attraction, its external surface, if it be nearly spherical, must be an ellipsoid of revolution.

This may be proved as follows:—

By Ex. 5, Art. 24, at the free surface of a liquid in relative equilibrium, if V denote the attraction potential, which in this case is a force function, and if the axis of rotation be taken as the axis of z, we have

$$dV + \omega^2(xdx + ydy) = 0;$$

whence, as in Art. 81, we get

$$V + \frac{\omega^2 r^2}{2}(1 - \mu^2) = \text{constant}. \tag{12}$$

The last term on the left-hand side of this equation must be small, as otherwise the surface of the liquid could not be approximately spherical. In this term, therefore, we may put $r = a$, and substituting for V from (7), we get

$$\frac{4}{3}\pi\rho a^2(1 - a\Sigma Y_n) + 4a\pi\rho a^2 \Sigma \frac{Y_n}{2n + 1} + \frac{\omega^2 a^2}{2}(1 - \mu^2) = \text{constant}. \tag{13}$$

In order to make use of this equation, we must express μ^2 by means of spherical harmonics. Since $\mu^2 = \dfrac{z^2}{r^2}$, it is plain that the solid harmonic corresponding to the spherical harmonic of highest degree in μ^2 must be $z^2 + kr^2$, where k is an undetermined constant. To determine k, we have

$$\nabla^2\{z^2 + k(x^2 + y^2 + z^2)\} = 0.$$

Hence $k = -\frac{1}{3}$, and $\mu^2 - \frac{1}{3} + \frac{1}{3}$ is the required expression for μ^2.

By Art. 139, the sum of the spherical harmonics of each degree above zero in (13) must vanish separately. Hence $Y_n = 0$ if $n > 2$, and

$$\frac{8}{15} a\pi\rho a^2 Y_2 = \frac{\omega^2 a^2}{2} (\tfrac{1}{3} - \mu^2).$$

Putting $\dfrac{\omega^2}{\frac{4}{3}\pi\rho} = q$, we get $aY_2 = \frac{5}{4} q (\frac{1}{3} - \mu^2)$.

Hence the equation of the free surface is of the form

$$r = a\{1 - \tfrac{5}{4} q\mu^2\},$$

which represents an ellipsoid of revolution nearly spherical whose ellipticity is $\frac{5}{4} q$. See Art. 81.

144. **Figure of the Earth.**—On the hypotheses that the Earth is composed of homogeneous layers bounded by similar surfaces nearly spherical, and that it is covered with liquid in relative equilibrium, it is easy to show that the external surface of the liquid must be an oblate ellipsoid of revolution whose axis is the axis of rotation.

The attraction potential V of the Earth is given by (11). At the surface of the liquid, (12) must hold good. Hence, by substitution, we obtain

$$\frac{4\pi}{a_1}(1 - a\Sigma Y_n)\int_0^{a_1} \rho a^2 da + 4a\pi \Sigma \frac{\int_0^{a_1} \rho d(a^{n+3} Y_n)}{(2n+1)a_1^{n+1}}$$
$$+ \frac{\omega^2 a_1^2}{2}(\tfrac{1}{3} - \mu^2) = \text{constant.} \quad (14)$$

Since the surfaces of equal density are similar, Y_n does not vary with a, and as a_1 is the greatest possible value for a if n be not less than 2, we have

$$a^2 > \frac{(n+3)\, a^{n+2}}{(2n+1)\, a_1^n}.$$

Hence the multiplier of Y_n in (14) cannot be zero, and therefore if $n > 2$, we have $Y_n = 0$.

If $n = 2$, we obtain

$$4\pi \left\{ a_1^2 \int_0^{a_1} \rho a^2 da - \int_0^{a_1} \rho a^4 da \right\} a Y_2 = \frac{\omega^2 a_1^5}{2} \left(\frac{1}{3} - \mu^2 \right). \quad (15)$$

By Art. 78, when the centre of inertia is the origin, the coefficient of $\frac{1}{r^2}$ in V is zero. Hence, in the expression for the potential of a spheroid given by (9), if the surfaces of equal density be similar, and if the centre of inertia be the origin, we must have $Y_1 = 0$; and in the present case the form of the external surface is determined by the equation $r = a_1(1 + a Y_2)$, where Y_2 is given by (15). The external surface is therefore an oblate ellipsoid of revolution having the axis of rotation as its axis.

It seems improbable that the hypothesis made above with respect to the form of the surfaces of equal density should be correct. In order that it should be true, it is necessary that these surfaces should have been formed under similar conditions; but, unless the Earth were of uniform density, this could not have been the case, since the equatorial centrifugal force due to rotation varies as the distance from the centre, whilst the attraction of the sphere having this distance as radius varies in a different manner unless the density be uniform.

A more probable hypothesis is, that the surfaces of equal density are represented by equations of the form

$$r = a(1 + ahy),$$

where h is a parameter varying with a, but constant for each surface, and y a function of μ and ϕ, which is the same for all the surfaces.

145. **Clairaut's Theorem.**—Whatever be the internal constitution of the Earth, if it be covered with liquid in relative equilibrium whose external surface is an ellipsoid of revolution nearly spherical, the ellipticity, ε, the ratio of the centrifugal force at the equator to gravity, q, and the difference between polar and equatorial gravity divided by the latter, γ, fulfil the relation $\gamma + \varepsilon = \frac{5}{2} q$.

This equation was proved in Art. 81 on a particular hypothesis as to the internal constitution of the Earth. Any hypothesis of this kind is, however, unnecessary, as was first pointed out by Sir G. Stokes.

At the external surface of the liquid, the Earth's potential V must satisfy (12); but as this surface is nearly spherical and the term in (12) due to rotation is small, the variable terms in V must be small. Hence, if M denote the mass of the Earth, we may assume

$$V = \frac{M}{r} + \alpha \sum \frac{a^n Y_n}{r^{n+1}},$$

where α is a small constant. Again, by Art. 81, the form of the external surface is represented by the equation

$$r = a(1 - \epsilon\mu^2) = a\left\{1 - \frac{\epsilon}{3} - \epsilon\left(\mu^2 - \frac{1}{3}\right)\right\}.$$

Substituting in (12), we get

$$\frac{M}{a}\{1 + \epsilon(\mu^2 - \tfrac{1}{3})\} + \alpha \sum \frac{Y_n}{a} + \frac{\omega^2}{2} a^2 (\tfrac{1}{3} - \mu^2) = \text{constant}.$$

Hence $Y_n = 0$, unless $n = 2$. If $n = 2$, we have

$$\alpha Y_2 = M\left(\frac{q}{2} - \epsilon\right)\left(\mu^2 - \frac{1}{3}\right),$$

where

$$q = \omega^2 a \bigg/ \frac{M}{a^2}.$$

Accordingly,

$$V = \frac{M}{r} + \alpha \frac{a^2 Y_2}{r^3} = \frac{M}{r} + \frac{Ma^2}{r^3}\left(\frac{q}{2} - \epsilon\right)\left(\mu^2 - \frac{1}{3}\right), \qquad (16)$$

and

$$-\frac{dV}{dr} = \frac{M}{r^2} + 3\alpha \frac{a^2 Y_2}{r^4}.$$

Hence, if g denote the acceleration due to gravity at any point on the Earth's surface, in the same manner as in Art. 81, we find

$$g = \frac{M}{a^2}(1 + 2\epsilon\mu^2) + \frac{3M}{a^2}\left(\frac{q}{2} - \epsilon\right)\left(\mu^2 - \frac{1}{3}\right) - \omega^2 a(1 - \mu^2);$$

that is,

$$g = \frac{M}{a^2}\left(1 + \varepsilon - \frac{3q}{2} + \left(\frac{5q}{2} - \epsilon\right)\mu^2\right\}. \qquad (17)$$

Hence

$$\gamma = \frac{g_p - g_e}{g_e} = \frac{5q}{2} - \varepsilon,$$

and therefore

$$\gamma + \varepsilon = \frac{5q}{2}.$$

146. **Tangential Component of Attraction.**—If P denote the component of the Earth's attraction perpendicular to the radius at any point on its surface, by (16), we have

$$P = \frac{dV}{rd\theta} = \frac{Ma^2}{r^4}(q - 2\varepsilon)\mu\frac{d\mu}{d\theta} = \frac{M}{a^2}\left(\epsilon - \frac{q}{2}\right)\sin 2\lambda, \qquad (18)$$

where λ denotes the latitude of the place.

If we compare (16) with (2), Art. 78, we get

$$Ma^2(2\varepsilon - q)(\mu^2 - \tfrac{1}{3}) = 3I - (A + B + C).$$

Hence $$C - A = \frac{Ma^2}{3}(2\epsilon - q). \qquad (19)$$

The equations proved above were arrived at before in Art. 81 by means of a special hypothesis with respect to the internal constitution of the Earth. The facility with which these results have been obtained in the present and preceding Articles without any such hypothesis illustrates the power of the Laplacian method.

147. **Legendre's Coefficients.**—The definition of these coefficients given in Art. 138 enables us to see that they are rational and integral functions of μ. A general expression for these coefficients cannot be readily obtained by the usual methods of expansion. If we put

$$(1 - 2yx + y^2)^{-\frac{1}{2}} = \frac{dz}{dx},$$

we can, by integration, get rid of the negative index; and thus we obtain

$$z = -\frac{(1 - 2yx + y^2)^{\frac{1}{2}}}{y} + \text{constant}.$$

If we take $\frac{1}{y}$ for the constant, we get

$$(yz - 1)^2 = 1 - 2yx + y^2;$$

whence we obtain $$z = x + \frac{y}{2}(z^2 - 1). \tag{20}$$

We have now an expression for z suitable for the application of Lagrange's theorem (Williamson, *Differential Calculus*, Art. 125) by which we obtain

$$z = x + y\frac{x^2 - 1}{2} + \frac{y^2}{1{\cdot}2}\frac{d}{dx}\left(\frac{x^2 - 1}{2}\right)^2 + \frac{y^3}{1{\cdot}2{\cdot}3}\frac{d^2}{dx^2}\left(\frac{x^2 - 1}{2}\right)^3 + \&c.;$$

whence

$$(1 - 2yx + y^2)^{-\frac{1}{2}} = \frac{dz}{dx} = 1 + y\frac{d}{dx}\left(\frac{x^2 - 1}{2}\right) + \frac{y^2}{1{\cdot}2}\frac{d^2}{dx^2}\left(\frac{x^2 - 1}{2}\right)^2 + \&c.$$

Hence, if $$(1 - 2\mu h + h^2)^{-\frac{1}{2}} = 1 + \Sigma P_n h^n,$$

we get $$P_n = \frac{1}{1{\cdot}2{\cdot}3\ldots n{\cdot}2^n}\frac{d^n}{d\mu^n}(\mu^2 - 1)^n. \tag{21}$$

The development of P_n in powers of μ is most easily effected by means of the differential equation satisfied by zonal harmonics.

If S_n denote a zonal harmonic of the degree n, we may assume

$$S_n = a_s \mu^s + a_{s-1} \mu^{s-1} + \&c.,$$

where s, &c., must be positive in order that S_n should be finite at every point of space, and S_n must satisfy the equation

$$\frac{d}{d\mu} (1 - \mu^2) \frac{dS_n}{d\mu} + n(n+1) S_n = 0. \qquad (22)$$

Hence we get

$$s(s-1) a_s \mu^{s-2} - s(s+1) a_s \mu^s - (s-2)(s-1) a_{s-2} \mu^{s-2} + \&c. \\ + n(n+1) \{a_s \mu^s + \&c.\} = 0.$$

Here the coefficient of μ^s is

$$\{n(n+1) - s(s+1)\} a_s,$$

and that of μ^{s-2} is

$$s(s-1) a_s + \{n(n+1) - (s-2)(s-1)\} a_{s-2};$$

accordingly, as each coefficient must vanish separately, we obtain

$$(s-n)(s+n+1) = 0, \quad a_{s-2} = \frac{-s(s-1)}{(n-s+2)(n+s-1)} a_s.$$

From the first of these we get $s = n$, or $s = -(n+1)$; and as the negative value for s is here inadmissible, we have

$$s = n, \quad a_{n-2} = \frac{-n(n-1)}{2(2n-1)} a_n, \quad a_{n-4} = \frac{-(n-2)(n-3)}{4(2n-3)} a_{n-2},$$

and in general

$$a_{n-2q} = (-1)^q \frac{n(n-1)(n-2)(n-3) \ldots (n-2q+2)(n-2q+1)}{2.4 \ldots 2q.(2n-1)(2n-3) \ldots (2n-2q+1)} a_n. \qquad (23)$$

It is plain that the terms in (22) resulting from

$$a_{s-1}\mu^{s-1} + a_{s-3}\mu^{s-3} + \&c.,$$

must vanish independently of those arising from the series already considered, and that we get for the first term the equation

$$(s-n-1)(s+n)a_{s-1} = 0.$$

Hence a_{s-1}, a_{s-3}, &c., must each be zero, and we obtain

$$S_n = a_n\left\{\mu^n - \frac{n(n-1)}{2(2n-1)}\mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}\mu^{n-4} - \&c.\right\}. \tag{24}$$

Hence, zonal harmonics of the same order can differ only in the constant factor, and we may write

$$S_n = aP_n, \tag{25}$$

where a is an undetermined constant.

It is easy to see that the coefficient of μ^n in (21) is

$$\frac{2n(2n-1)(2n-2)\ldots(n+1)}{2^n\lfloor n} \quad \text{or} \quad \frac{2^n.\lfloor n.1.3.5\ldots(2n-1)}{2^n.\lfloor n.\lfloor n},$$

and therefore

$$\left.\begin{aligned} P_n &= \frac{1}{2^n\lfloor n}\frac{d^n}{d\mu^n}(\mu^2-1)^n \\ &= \frac{(n+1)(n+2)\ldots 2n}{2^n\lfloor n}\left\{\mu^n - \frac{n(n-1)}{2(2n-1)}\mu^{n-2} + \&c.\right\} \\ &= \frac{1.3.5.(2n-1)}{\lfloor n}\{\mu^n - \&c.\} \end{aligned}\right\}. \tag{26}$$

It follows from the definition of P_n that when $\mu = 1$ the value of P_n is unity.

148. **Spherical Harmonics.**—Since the expansion of

$$\left(1 - 2\frac{xx' + yy' + zz'}{r'^2} + \frac{r^2}{r'^2}\right)^{-\frac{1}{2}}$$

contains only rational and integral functions of x, y, and z, the coefficients L_1, L_2, &c., must be rational and integral functions of $\sin\phi$ and $\cos\phi$, in which each power of $\sin\phi$ and $\cos\phi$ is multiplied by the same power of $\sqrt{(1-\mu^2)}$. Hence, as

$$\frac{4\pi}{2n+1} Y_n = \int_{-1}^{+1}\int_0^{2\pi} L_n Y'_n \, d\mu' \, d\phi',$$

the spherical harmonic Y_n must be a rational and integral function of $\sin\phi$ and $\cos\phi$ of the n^{th} degree in which each power of $\sin\phi$ and $\cos\phi$ is multiplied by the same power of $\sqrt{(1-\mu^2)}$. If each power of $\sin\phi$ and $\cos\phi$ be expanded in a series of sines and cosines of multiples of ϕ, we see that finally Y_n is reducible to the form

$$\Sigma\,(A_s M_s \cos s\phi + B_s N_s \sin s\phi),$$

where A_s and B_s are undetermined constants, and M_s and N_s functions of μ.

If we put $\dfrac{d}{d\mu} = D$, and $\mu^2 - 1 = u$, equation (1) becomes

$$DuDY_n + \frac{1}{u}\frac{d^2 Y_n}{d\phi^2} - n(n+1)Y_n = 0. \qquad (27)$$

Since the coefficient of the sine or cosine of each multiple of ϕ must vanish separately in (27), we have

$$DuDM_s - \left\{\frac{s^2}{u} + n(n+1)\right\} M_s = 0. \qquad (28)$$

Again, since $\cos s\phi$ and $\sin s\phi$ can result only from

$$(\cos\phi)^s, \quad (\sin\phi)^s, \quad (\cos\phi)^{s+2}, \quad (\sin\phi)^{s+2}, \quad \text{\&c.},$$

M_s must contain $u^{\frac{s}{2}}$ as a factor; and the other factor must be

a rational and integral function of μ. Accordingly, we may put

$$M_s = u^{\frac{s}{2}}v,$$

where v denotes a rational and integral function of μ. From (28), we have then

$$s^2\mu^2 u^{\frac{s}{2}-1} v + su^{\frac{s}{2}} v + (2s+2) u^{\frac{s}{2}} \mu Dv + u^{\frac{s}{2}+1} D^2 v$$

$$- s^2 u^{\frac{s}{2}-1} v - n(n+1) u^{\frac{s}{2}} v = 0. \qquad (29)$$

Since $u = \mu^2 - 1$, equation (29) is divisible by $u^{\frac{s}{2}}$, and we get

$$uD^2v + (s+1) Du\, Dv + \frac{s(s+1)}{2} vD^2u - n(n+1)v = 0. \qquad (30)$$

If we assume $v = D^s w$, equation (30) becomes

$$D^{s+1}(uDw) - n(n+1)D^s w = 0. \qquad (31)$$

Since v is a rational and integral function of μ, it is plain that, with the exception of a constant factor, it is completely determined by (30). Hence any rational and integral function of μ which satisfies (30) or (31) must represent v. Equation (31) is satisfied if w satisfy

$$Du\, Dw - n(n+1)w = 0;$$

but this equation is the same as (22).

Hence we may put $w = P_n$, and we have

$$v = D^s P_n, \quad M_s = u^{\frac{s}{2}} D^s P_n.$$

It is plain that the equations by which N_s is determined are the same as those for M_s. Accordingly, these two functions can differ only by a constant factor, and we obtain

$$Y_n = \Sigma (A_s \cos s\phi + B_s \sin s\phi) u^{\frac{s}{2}} D^s P_n. \qquad (32)$$

The part of Y_n depending upon $s\phi$, that is,

$$(A_s \cos s\phi + B_s \sin s\phi)\, u^{\frac{s}{2}} D^s P_n,$$

is termed a *tesseral harmonic of degree n and order s*, and we may write

$$Y_n = \Sigma\, T_{ns}(A_s \cos s\phi + B_s \sin s\phi). \qquad (33)$$

If we substitute for P_n its value given by (26), since A_s and B_s are undetermined constants, we have

$$T_{ns} = (1-\mu^2)^{\frac{s}{2}}$$

$$\times \left\{ \mu^{n-s} - \frac{(n-s)(n-s-1)}{2(2n-1)}\mu^{n-s-2} + \frac{(n-s)(n-s-1)(n-s-2)(n-s-3)}{2.4.(2n-1)(2n-3)}\mu^{n-s-4} - \&c. \right\}. \qquad (34)$$

149. **Laplace's Coefficients.**—Laplace's coefficients are, as we have seen in Art. 138, a particular kind of spherical harmonic; and as they are functions of ϕ and ϕ' through being functions of $\cos(\phi - \phi')$, and are symmetrical in μ and μ', we must have

$$L_n = \Sigma a_s \cos s(\phi - \phi')\, u^{\frac{s}{2}} u'^{\frac{s}{2}} D^s P_n D'^s P'_n, \qquad (35)$$

where a_s is a *definite function* of n and s, which may be determined in the following manner:—

By (4) we have

$$\int_{-1}^{+1}\int_0^{2\pi} \left\{ \Sigma a_s (\cos s\phi \cos s\phi' + \sin s\phi \sin s\phi')\, u^{\frac{s}{2}} u'^{\frac{s}{2}} D^s P_n D'^s P'_n \right\}$$

$$\times \left\{ \Sigma (A_s \cos s\phi + B_s \sin s\phi)\, u^{\frac{s}{2}} D^s P_n \right\}$$

$$\times\, d\mu\, d\phi = \frac{4\pi}{2n+1} \Sigma (A_s \cos s\phi' + B_s \sin s\phi')\, u'^{\frac{s}{2}} D'^s P'_n.$$

It is plain that the only part of the multiplier of $\cos s\phi'$ in the left-hand member of this equation which does not vanish after integration is

$$a_s A_s u'^{\frac{s}{2}} D'^s P'_n \int_{-1}^{+1} \int_0^{2\pi} u^s (D^s P_n)^2 \cos^2 s\phi \, d\mu \, d\phi.$$

Hence, if s be not zero, we get

$$a_s \int_{-1}^{+1} u^s (D^s P_n)^2 d\mu = \frac{4}{2n+1}; \tag{36}$$

and if $s = 0$, we have

$$a_0 \int_{-1}^{+1} P^2_n \, d\mu = \frac{2}{2n+1}. \tag{37}$$

The first term in L_n is $a_0 P_n P'_n$, and when $\mu' = 1$, all the other terms vanish, and $P'_n = 1$. Hence, in this case, $L_n = a_0 P_n$; but L_n becomes P_n when $\mu' = 1$, and therefore $a_0 = 1$. The remaining coefficients can now be found by means of (36) and (37).

Let $$\int_{-1}^{+1} u^s (D^s P_n)^2 d\mu = \Lambda_s,$$

then $$\Lambda_1 = \int u (DP_n)^2 d\mu;$$

also by (2), we have

$$n(n+1)\Lambda_0 = \int P_n D(uDP_n) \, d\mu;$$

whence

$$\Lambda_1 + n(n+1)\Lambda_0 = \int_{-1}^{+1} \{P_n D(uDP_n) + uDP_n DP_n\} \, d\mu$$

$$= \int_{-1}^{+1} D(uP_n DP_n) d\mu = 0,$$

since u vanishes at each limit of the integral.

It is now easy to see that an equation similar to that obtained above holds good for any two successive integrals of the series. In fact, by (2), we have

$$D^{s+1}(uDP_n) - n(n+1)D^sP_n = 0;$$

whence, remembering that $D^2u = 2$, $D^3u = 0$, by Leibnitz's theorem, we have

$$uD^{s+2}P_n + (s+1)Du\,D^{s+1}P_n + s(s+1)D^sP_n = n(n+1)D^sP_n,$$

and therefore

$$(n-s)(n+s+1)u^sD^sP_n = D(u^{s+1}D^{s+1}P_n). \tag{38}$$

Hence we have

$$\begin{aligned}\Lambda_{s+1} &+ (n-s)(n+s+1)\Lambda_s \\ &= \int_{-1}^{+1}\{(u^{s+1}D^{s+1}P_n)DD^sP_n + D^sP_nD(u^{s+1}D^{s+1}P_n)\}d\mu \\ &= \int_{-1}^{+1}D(u^{s+1}D^sP_n\,D^{s+1}P_n)\,d\mu = 0. \end{aligned} \tag{39}$$

Accordingly, $\Lambda_{s+1} = -(n-s)(n+s+1)\Lambda_s$; (40)

and therefore, by (36), if s be not zero, we have

$$a_{s+1} = -\frac{a^s}{(n-s)(n+s+1)}; \tag{41}$$

and, by (37), we get

$$a_1 = \frac{-2a_0}{n(n+1)}. \tag{42}$$

Hence, as $a_0 = 1$, we obtain

$$a_1 = \frac{-2}{n(n+1)}, \ldots, \quad a_s = \frac{2(-1)^s}{n(n+1)\ldots(n+s)(n-1)(n-2)\ldots(n-s+1)},$$

that is,
$$a_s = \frac{2(-1)^s\,\lfloor n-s}{\lfloor n+s}, \tag{43}$$

and

$$L_n = P_nP'_n + 2\Sigma(-1)^s(1-\mu^2)^{\frac{s}{2}}(1-\mu'^2)^{\frac{s}{2}}D^sP_nD'^sP'_n\frac{\lfloor n-s}{\lfloor n+s}\cos s(\phi-\phi'). \tag{44}$$

150. Complete Harmonics.—The definition of solid and spherical harmonics in general has been given in Art. 137; but the properties of spherical harmonics proved in Art. 139 have been obtained on the hypothesis that these functions are finite and single-valued for every point of space, and in that Art. m and n denote integers.

If Y_i denote a function of μ and ϕ satisfying the equation

$$\frac{d}{d\mu}(1-\mu^2)\frac{dY_i}{d\mu} + \frac{1}{1-\mu^2}\frac{d^2Y_i}{d\phi^2} + i(i+1)Y_i = 0, \qquad (45)$$

where i denotes any real numerical quantity, corresponding to Y_i, there are two solid harmonics, viz. $r^i Y_i$ and $r^{-(i+1)} Y_i$.

As i is real, one of the quantities i and $-(i+1)$ is negative.

Hence, selecting the two solid harmonics of negative degree which correspond to Y_i and Y_j, we see that when Y_i and Y_j are finite and single-valued, equation (3) holds good, unless $i = j$, or $i = -(j+1)$. Again, if Y_i be finite and single-valued, by a process similar to that employed in proving (4), Art. 139, we have

$$\int_{-1}^{+1}\int_0^{2\pi} Y_i \Sigma (i+n+1) L_n \left(\frac{a}{r'}\right)^{n+1} d\mu\, d\phi = 4\pi Y'_i \left(\frac{a}{r'}\right)^{i+1}, \quad (46)$$

if i be positive, the coefficient of L_n being $n-i$ if i be negative, and that of Y'_i being $4\pi\left(\dfrac{r'}{a}\right)^i$.

Accordingly, by (3), $Y'_i = 0$, unless $i = n$, or $i = -(n+1)$. In either case Y_i is a rational and integral function of μ,

$$\sqrt{1-\mu^2}\cos\phi, \quad \text{and} \quad \sqrt{1-\mu^2}\sin\phi$$

of the degree n.

Hence we conclude that the degree of a complete spherical harmonic must be a *positive integer*, and that the corresponding solid harmonic of positive degree must be a rational and integral function of x, y, and z.

This last result is usually expressed by saying that every complete solid harmonic is a rational and integral function of x, y, and z, or can be made so by multiplying by a suitable power of r.

151. Reduction of a Function to Spherical Harmonics.—It was shown in Art. 139 that a finite and single-valued function of μ and ϕ can always be expressed by a series of complete spherical harmonics. If this series be finite, so that

$$f(\mu\phi) = Y_0 + Y_1 \ldots + Y_n,$$

we have

$$r^n f = V_n + r^2 V_{n-2} + \&c. + r\{V_{n-1} + r^2 V_{n-3} + \&c.\}.$$

Hence $r^n f = f_n + r f_{n-1}$ where f_n and f_{n-1} denote rational, integral, homogeneous functions of the coordinates; and it appears that if a function of μ and ϕ can be expressed by a finite series of spherical harmonics, the corresponding function of the coordinates must consist of a rational, integral, homogeneous function, together with another such function multiplied by r. Accordingly, the problem to express a given function of μ and ϕ in a finite series of spherical harmonics, when soluble, is reduced to that of expressing f_n, a rational, integral, homogeneous function of the coordinates in a series of the form $V_n + r^2 V_{n-2} + \&c.$

This is effected most easily by means of Laplace's operator.

In fact, by Leibnitz's theorem,

$$\nabla^2 r^p V_m = V_m \nabla^2 r^p + 2pr^{p-1} \times \left(\frac{dr}{dx}\frac{dV_m}{dx} + \frac{dr}{dy}\frac{dV_m}{dy} + \frac{dr}{dz}\frac{dV_m}{dz}\right) + r^p \nabla^2 V_m, \tag{47}$$

but $$\frac{d^2}{dx^2} r^p = p\frac{d}{dx} x r^{p-2} = p\{r^{p-2} + (p-2)x^2 r^{p-4}\},$$

and $\nabla^2 r^p = p(p+1)r^{p-2}$; also, $\nabla^2 V_m = 0$, and

$$x\frac{dV_m}{dx} + y\frac{dV_m}{dy} + z\frac{dV_m}{dz} = mV_m.$$

Accordingly,

$$\nabla^2 r^p V_m = \{p(p+1) + 2pm\} r^{p-2} V_m. \tag{48}$$

From (48), we get

$$\nabla^2 f_n = a_2 V_{n-2} + a_4 r^2 V_{n-4} + \&c., \quad \nabla^4 f_n = b_4 V_{n-4} + b_6 r^2 V_{n-6} + \&c.,$$
$$\&c., \qquad\qquad \&c.,$$

where a_2, a_4, &c., b_4, &c., are known numerical coefficients. If this process be repeated sufficiently often, we find ultimately

$$\nabla^{2q} f_n = h V_0, \quad \text{or} \quad \nabla^{2q} f_n = k V_1,$$

according as $n = 2q$, or $n = 2q + 1$, the coefficients h and k being known numbers. In fact,

$$h = \lfloor n + 1, \quad \text{and} \quad k = \frac{n+2}{3} \lfloor n.$$

By the equations previously obtained, we can then determine the other solid harmonics.

As a simple example, let

$$f = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy,$$

then $f = V_2 + r^2 V_0$, and $\nabla^2 f = h V_0$, where $h = 3 . 2 . 1 = 6$;

but $\nabla^2 f = 2(A + B + C)$; hence $V_0 = \frac{1}{3}(A + B + C)$,

$$V_2 = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy$$
$$- \tfrac{1}{3}(A + B + C)(x^2 + y^2 + z^2),$$

and

$$f = \tfrac{1}{3}\{(2A - B - C)x^2 + (2B - C - A)y^2 + (2C - A - B)z^2\}$$
$$+ 2Fyz + 2Gzx + 2Hxy + \tfrac{1}{3}(A + B + C)r^2.$$

Again, let $f = x^3 + y^3 + x^2y + y^2x$,

then $f = V_3 + r^2 V_1$, and $\nabla^2 f = k V_1$, where $k = \frac{5}{3} . 3 . 2 = 10$;

accordingly, $10 V_1 = 8(x + y)$, hence $V_1 = \frac{4}{5}(x + y)$, and

$$V_3 = \tfrac{1}{5}(x^3 + y^3 + x^2y + y^2x) - \tfrac{4}{5}z^2(x + y).$$

The method originally given by Laplace for reducing to a series of spherical harmonics a function of μ and ϕ corresponding to a rational and integral function of the coordinates, differs somewhat from that given above.

A rational and integral function of the coordinates corresponds to a rational and integral function of

$$\mu, \quad \sqrt{1-\mu^2}\cos\phi, \quad \text{and} \quad \sqrt{1-\mu^2}\sin\phi.$$

If the various powers of $\cos\phi$ and $\sin\phi$ be developed in sines and cosines of multiples of ϕ, the series multiplying

$$(1-\mu^2)^{\frac{s}{2}}\cos s\phi$$

will contain all the powers of μ not exceeding $n-s$, where n is the degree of the given function of the coordinates.

If we collect together the terms containing the highest power of μ in each series, we obtain an expression of the form

$$\Sigma(A_s\cos s\phi + B_s\sin s\phi)(1-\mu^2)^{\frac{s}{2}}\mu^{n-s} + A_0\mu^n,$$

the function Y_n may then be determined by taking its $2n+1$ arbitrary constants, so that the terms of the above form may be equal to those in the expression given above. If we subtract Y_n thus determined from f, we get a function, $f-Y_n$ of the degree $n-1$ in

$$\mu, \quad \sqrt{1-\mu^2}\cos\phi, \quad \text{and} \quad \sqrt{1-\mu^2}\sin\phi.$$

The harmonic Y_{n-1} can then be determined in a way similar to that employed in finding Y_n, and so on.

When the original function of the coordinates is transformed into a function of r, μ, and ϕ, the various powers of r are in f regarded as constants.

It is plain that the total number of terms or of independent constants in f is $1+3+5\ldots+2n+1$, that is, $(n+1)^2$.

This is also the number of arbitrary constants in the series

$$Y_0 + Y_1 \ldots + Y_n.$$

152. **Methods of forming Complete Solid Harmonics.**—A complete solid harmonic of positive degree is, as we have seen, Art. 150, a rational and integral function of the coordinates. A solid harmonic of the degree n, since it is homogeneous, contains, therefore, $\dfrac{(n+1)(n+2)}{2}$ terms.

The coefficients of these terms are, however, not all independent; for, if V_n denote the harmonic, $\nabla^2 V_n$ must vanish for all values of the coordinates, and therefore $\frac{(n-1)n}{2}$ equations must be satisfied by the coefficients of V_n.

Accordingly, V_n contains $2n+1$ independent arbitrary constants.

Since $\nabla^2\left(\frac{1}{r}\right) = 0$, we have $\left(\frac{d}{dx}\right)^i \left(\frac{d}{dy}\right)^j \left(\frac{d}{dz}\right)^k \nabla^2 \frac{1}{r} = 0$,

where i, j, k denote any integers.

Hence $$\nabla^2 \left(\frac{d}{dx}\right)^i \left(\frac{d}{dy}\right)^j \left(\frac{d}{dz}\right)^k \frac{1}{r} = 0,$$

and therefore $$\left(\frac{d}{dx}\right)^i \left(\frac{d}{dy}\right)^j \left(\frac{d}{dz}\right)^k \frac{1}{r}$$

is a solid harmonic of the degree $-(i+j+k+1)$. If $i+j+k=n$, the number of different combinations of the type
$$\left(\frac{d}{dx}\right)^i \left(\frac{d}{dy}\right)^j \left(\frac{d}{dz}\right)^k$$

which can be formed is $\frac{(n+1)(n+2)}{2}$; but all the different functions which result by the use of these operators on $\frac{1}{r}$ are not independent. In fact,

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right) \left(\frac{d}{dx}\right)^{i'} \left(\frac{d}{dy}\right)^{j'} \left(\frac{d}{dz}\right)^{k'} \frac{1}{r} = 0,$$

where $i'+j'+k' = n-2$. There are $\frac{(n-1)n}{2}$ equations of this form which must be satisfied identically by functions of the form
$$\left(\frac{d}{dx}\right)^i \left(\frac{d}{dy}\right)^j \left(\frac{d}{dz}\right)^k \frac{1}{r},$$

where $i + j + k = n$. Consequently, of these latter functions there are only $2n + 1$ independent. Hence, every complete solid harmonic V_n of the degree n is given by the equation

$$V_n = r^{2n+1} \Sigma A_{ijk} \left(\frac{d}{dx}\right)^i \left(\frac{d}{dy}\right)^j \left(\frac{d}{dz}\right)^k \frac{1}{r}, \tag{49}$$

where $i + j + k = n$, and where there are $2n + 1$ independent functions, and consequently $2n + 1$ independent arbitrary constants.

Another method of forming complete solid harmonics depends on the consideration that, if a_1, β_1, γ_1 be the direction cosines of any line,

$$\left(a_1 \frac{d}{dx} + \beta_1 \frac{d}{dy} + \gamma_1 \frac{d}{dz}\right) \frac{1}{r}$$

satisfies Laplace's equations, and more generally that this equation is satisfied by

$$\left(a_1 \frac{d}{dx} + \beta_1 \frac{d}{dy} + \gamma_1 \frac{d}{dz}\right)\left(a_2 \frac{d}{dx} + \beta_2 \frac{d}{dy} + \gamma_2 \frac{d}{dz}\right) \dots \times \left(a_n \frac{d}{dx} + \beta_n \frac{d}{dy} + \gamma_n \frac{d}{dz}\right) \frac{1}{r}.$$

It follows from this that the function

$$A r^{2n+1} \left(a_1 \frac{d}{dx} + \beta_1 \frac{d}{dy} + \gamma_1 \frac{d}{dz}\right)\left(a_2 \frac{d}{dx} + \beta_2 \frac{d}{dy} + \gamma_2 \frac{d}{dz}\right) \dots \times \left(a_n \frac{d}{dx} + \beta_n \frac{d}{dy} + \gamma_n \frac{d}{dz}\right) \frac{1}{r}$$

satisfies Laplace's equation; and as it is a rational, integral, homogeneous function of the n^{th} degree, containing $2n + 1$ independent arbitrary constants, every complete solid harmonic of the n^{th} degree can be expressed in this form.

It is not, however, obvious that a set of *real* values of the coefficients a_1, β_1, γ_1, &c., corresponding to any given complete solid harmonic always exists, and that in general there is only one such set.

This proposition, which is necessary to complete Maxwell's method of representing solid harmonics, was proved by Sylvester (*Phil. Mag.*, October, 1876), in the following manner:—

It has been shown above, that by the solution of linear equations for determining the coefficients, we can reduce any complete solid harmonic to the form given by (49).

We have now to show that any rational homogeneous function of the n^{th} degree of the symbols of differentiation operating on $\frac{1}{r}$ can be reduced to the product of n real linear factors of the form

$$a\frac{d}{dx} + \beta\frac{d}{dy} + \gamma\frac{d}{dz}.$$

Since the symbols of differentiation obey the same laws as quantities, and since

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\frac{1}{r} = 0,$$

the theorem just stated is equivalent to asserting that any ternary quantic $(x, y, z)^n$, whose variables are subject to the condition $x^2 + y^2 + z^2 = 0$, can be reduced to the product of n real linear factors, and that this reduction can be effected in only one way.

The equations $(x, y, z)^n = 0$, $x^2 + y^2 + z^2 = 0$ may be regarded as representing plane curves having $2n$ points of intersection. If these points be joined in pairs, we obtain n straight lines, the coordinates of whose points of determination are obtained by solving for $x : y : z$ the simultaneous equations $(x, y, z)^n = 0$, and $x^2 + y^2 + z^2 = 0$.

If $\frac{x}{z}$ be real, the corresponding value of $\frac{y}{z}$ given by the latter equation must be imaginary, and so also, therefore, that of $\frac{x}{y}$. Hence of the three ratios, $x : y : z$, two at least are imaginary.

The equation of the straight line joining the points x', y', z', and x'', y'', z'' is

$$x(y'z'' - z'y'') + y(z'x'' - x'z'') + z(x'y'' - y'x'') = 0.$$

If we suppose $\frac{y'}{z'}$ and $\frac{x'}{z'}$ to be each imaginary, and select for $\frac{y''}{z''}$ and $\frac{x''}{z''}$, the conjugate imaginaries, each term in the equation of the straight line contains $\sqrt{-1}$ as a factor, and the line is therefore real.

If the equation of the degree $2n$ for determining $\frac{y}{z}$ have $2m$ imaginary roots, there are $2m$ imaginary values of either $\frac{x}{z}$ or $\frac{x}{y}$ corresponding, and therefore m real straight lines. Corresponding to the $2(n-m)$ real values of $\frac{y}{z}$, there must be $2(n-m)$ imaginary values of $\frac{x}{z}$ and $\frac{x}{y}$, and therefore $(n-m)$ additional real straight lines. Hence in all there are n real straight lines passing through the points of intersection of $(x, y, z)^n = 0$ and $x^2 + y^2 + z^2 = 0$.

There are no more. For if we seek the values of $\frac{y}{z}$ which satisfy the equation of a real straight line, and the equation $x^2 + y^2 + z^2 = 0$, these values must be real, or else conjugate imaginaries; and in the former case, the values of $\frac{x}{z}$ must be conjugate imaginaries, and also those of $\frac{x}{y}$. Hence, to obtain a real straight line, each imaginary value of one of the ratios $\frac{y}{z}$, &c., satisfying $(x, y, z)^n = 0$ and $x^2 + y^2 + z^2 = 0$, must be combined with its conjugate; consequently there are only n such lines.

Let $L = 0$ denote the equation of n straight lines passing through the $2n$ points of intersection of $(x, y, z)^n = 0$, and $x^2 + y^2 + z^2 = 0$; then, whenever $(x, y, z)^n$ and $x^2 + y^2 + z^2$ both vanish, so must L, and therefore

$$L = X(x, y, z)^n + Y(x^2 + y^2 + z^2). \qquad (50)$$

From the degree of the various functions in this equation we see that X is constant, and Y of the degree $n-2$. Since, in general, a ternary quantic of the n^{th} degree contains $\frac{(n+1)(n+2)}{2}$ constants, and the equation of n straight lines contains $2n+1$ constants; and since

$$\frac{(n+1)(n+2)}{2} = 2n+1+\frac{(n-1)n}{2},$$

it is plain that the $\frac{(n-1)n}{2}$ constants of Y can be so determined that the right-hand side of (50) shall represent n straight lines. It has been proved above that for one of these determinations the n straight lines are real. If $\alpha_1 x + \beta_1 y + \gamma_1 z = 0$, &c., represent these real lines, then

$$\begin{aligned}(\alpha_1 x + \beta_1 y + \gamma_1 z) \ldots (\alpha_n x + \beta_n y + \gamma_n z) \\ = A(x, y, z)^n + Y(x^2 + y^2 + z^2).\end{aligned}$$

Applying the theorem which has been proved for the quantities x, y, z to the symbols of differentiation, by (49), we get

$$\left.\begin{aligned} V_n &= r^{2n+1}\left\{\Sigma A_{ijk}\left(\frac{d}{dx}\right)^i\left(\frac{d}{dy}\right)^j\left(\frac{d}{dz}\right)^k \right. \\ &\left. + \Sigma L_{i'j'k'}\left(\frac{d}{dx}\right)^{i'}\left(\frac{d}{dy}\right)^{j'}\left(\frac{d}{dz}\right)^{k'}\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\right\}\frac{1}{r} \\ &= Ar^{2n+1}\left(\alpha_1\frac{d}{dx}+\beta_1\frac{d}{dy}+\gamma_1\frac{d}{dz}\right)\ldots \\ &\times\left(\alpha_n\frac{d}{dx}+\beta_n\frac{d}{dy}+\gamma_n\frac{d}{dz}\right)\frac{1}{r}\end{aligned}\right\}, \quad (51)$$

where V_n denotes any solid harmonic of degree n, A, A_{ijk}, and $L_{i'j'k'}$ constants, and $\alpha_1, \beta_1, \gamma_1$, &c., the direction cosines of straight lines, and where

$$i+j+k=n, \quad i'+j'+k'=n-2.$$

If lines be drawn from the origin, each in one direction, having α_1, β_1, γ_1, &c., as their direction cosines, these lines meet a sphere, having the origin for centre, in n points which are called the *poles of the corresponding spherical harmonic.*

The mode now described of forming spherical harmonics was given by Clerk Maxwell in his treatise on *Electricity and Magnetism.*

Maxwell's method of representing spherical and solid harmonics admits of an interesting physical interpretation.

If h_1 denote a line whose direction cosines are α_1, β_1, γ_1 drawn through the point x, y, z, and h'_1 the parallel line through the origin,

$$\left(\alpha_1 \frac{d}{dx} + \beta_1 \frac{d}{dy} + \gamma_1 \frac{d}{dz}\right) \frac{1}{r} = \frac{d}{dh_1} \frac{1}{r} = -\frac{d}{dh'_1} \frac{1}{r},$$

and
$$V_n = Ar^{2n+1} \frac{d}{dh_1} \frac{d}{dh_2} \cdots \frac{d}{dh_n} \frac{1}{r}. \qquad (52)$$

Again, $\frac{m}{r}$ expresses the potential of a mass m at the origin, and

$$-\,m dh'_1 \frac{d}{dh'_1} \frac{1}{r}$$

expresses the potential produced by superimposing on this mass another negative mass of equal magnitude, situated at a point at a distance from the origin infinitely small in the direction h'_1. If this system be displaced through the distance dh'_2, reversed, and superimposed on the former, the potential becomes

$$m\, dh'_1\, dh'_2 \frac{d}{dh'_1} \frac{d}{dh'_2} \frac{1}{r}, \quad \text{and so on.}$$

The repetition of this process n times leads to the potential U_n, where

$$U_n = (-1)^n M \frac{d}{dh'_1} \frac{d}{dh'_2} \cdots \frac{d}{dh'_n} \frac{1}{r}$$
$$= M \frac{d}{dh_1} \frac{d}{dh_2} \cdots \frac{d}{dh_n} \frac{1}{r} = \frac{V_n}{r^{2n+1}},$$

provided
$$A = m\, dh_1\, dh_2 \ldots dh_n = M.$$

If A be a finite constant, m must be an infinitely great quantity of the n^{th} order.

As an easy example, illustrating the foregoing theory, we may consider the question to express in Maxwell's form a solid harmonic of the second degree containing only the squares of the variables.

Here, by Art. 151, the solid harmonic

$$V_2 = ax^2 + by^2 - (a+b)z^2.$$

Again, as $\frac{d^2}{dx^2} + \frac{d^2}{dy^2} = -\frac{d^2}{dz^2}$, we have

$$\left(A\frac{d^2}{dx^2} + B\frac{d^2}{dy^2} + C\frac{d^2}{dz^2}\right)\frac{1}{r} = \left((A-C)\frac{d^2}{dx^2} + (B-C)\frac{d^2}{dy^2}\right)\frac{1}{r},$$

and therefore $$V_2 = r^5\left(\lambda\frac{d^2}{dx^2} + \mu\frac{d^2}{dy^2}\right)\frac{1}{r}$$

$$= 3\lambda x^2 + 3\mu y^2 - (\lambda+\mu)r^2 = (2\lambda-\mu)x^2 + (2\mu-\lambda)y^2 - (\lambda+\mu)z^2.$$

Hence, comparing with the former expression for V_2, we have $2\lambda - \mu = a$, $2\mu - \lambda = b$; whence

$$\lambda = \tfrac{2}{3}a + \tfrac{1}{3}b, \quad \mu = \tfrac{1}{3}a + \tfrac{2}{3}b.$$

To reduce $\left(\lambda\frac{d^2}{dx^2} + \mu\frac{d^2}{dy^2}\right)\frac{1}{r}$ to Maxwell's form, we must consider the relative values of λ and μ.

If λ and μ have different algebraic signs, and $\mu = -\mu'$, then

$$V_2 = r^5\left(\sqrt{\lambda}\frac{d}{dx} + \sqrt{\mu'}\frac{d}{dy}\right)\left(\sqrt{\lambda}\frac{d}{dx} - \sqrt{\mu'}\frac{d}{dy}\right)\frac{1}{r}.$$

If λ and μ have the same sign and λ be the greater,

$$V_2 = r^5\left\{\lambda\frac{d^2}{dx^2} + \mu\frac{d^2}{dy^2} - \mu\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\right\}\frac{1}{r}$$

$$= r^5\left(\sqrt{\lambda-\mu}\frac{d}{dx} + \sqrt{\mu}\frac{d}{dz}\right)\left(\sqrt{\lambda-\mu}\frac{d}{dx} - \sqrt{\mu}\frac{d}{dz}\right)\frac{1}{r}.$$

The most general solid harmonic of the second degree is reducible to the form considered above by a transformation of the axes of coordinates.

In general, for the second degree, the reduction of a solid harmonic to Maxwell's form is mathematically the same problem as the determination of the planes of circular section of a quadric surface.

153. **Incomplete Harmonics.**—We have seen, Art. 150, that if Y_i be a *complete* spherical harmonic whose degree is real, i must be a positive integer, and Y_i a rational and integral function of μ, $\sqrt{(1-\mu^2)}\cos\phi$, and $\sqrt{(1-\mu^2)}\sin\phi$.

If i be a negative integer, $-(i+1)$ is zero or a positive integer.

If i be real but not an integer, it is easy to obtain expressions for Y_i which satisfy the differential equation (45); but these expressions become infinite at certain points on a sphere surrounding the origin, or alter in value after having passed continuously through a complete circuit surrounding the axis of z. In the latter case, accordingly, they are not single-valued.

If we assume $\quad p_i = a_0 + a_2\mu^2 + a_4\mu^4 + \&c.,$

$$q_i = a_1\mu + a_3\mu^3 + a_5\mu^5 + \&c.,$$

and substitute p_i in the equation

$$\frac{d}{d\mu}(1-\mu^2)\frac{dp}{d\mu} + i(i+1)p = 0, \tag{53}$$

in order that each power of μ should vanish, we find that

$$a_{n+2} = -\frac{(i-n)(i+n+1)}{(n+1)(n+2)}a_n. \tag{54}$$

Equation (54) is fulfilled also by two successive coefficients in the series denoted by q_i, provided q_i satisfies (53). Hence, if we assume

$$\left.\begin{aligned} p_i &= a_0\left(1 - \frac{i(i+1)}{1.2}\mu^2 + \frac{i(i-2)(i+1)(i+3)}{1.2.3.4}\mu^4 - \&c.\right), \\ q_i &= a_1\left(\mu - \frac{(i-1)(i+2)}{2.3}\mu^3 + \frac{(i-1)(i-3)(i+2)(i+4)}{2.3.4.5}\mu^5 - \&c.\right), \end{aligned}\right\} \tag{55}$$

each of the series p_i and q_i satisfies the differential equation for a spherical harmonic of the degree i, whatever be the value of i.

If i be an integer, one of these series terminates: the other contains an infinite number of terms.

If i be not an integer, both series contain an infinite number of terms.

The sum of each series is finite so long as $\mu < 1$; but if $\mu = 1$, either series, if it contains an infinite number of terms, becomes infinite. In fact, (54) may be written

$$a_{n+2} = \frac{(n-i)(n+i+1)}{(n+1)(n+2)} a_n;$$

and, accordingly, as n increases without limit, all the terms become of the same algebraical sign, and the value of $\frac{a_{n+2}}{a_n}$ tends to become unity. Hence (Williamson, *Differential Calculus*, Art. 73), if $\mu < 1$, the series is convergent.

In the case of the more general spherical harmonic Y_i, whatever be the value of i, we may assume

$$Y_i = \Sigma (A_s \cos s\phi + B_s \sin s\phi)(1-\mu^2)^{\frac{s}{2}} p_{is};$$

then, as in Art. 148, equation (30), we find that p_{is} must satisfy the equation

$$(\mu^2 - 1) D^2 p + 2\mu (s+1) Dp - (i-s)(i+s+1) p = 0. \quad (56)$$

This equation is satisfied by the series

$$a_0 + a_2\mu^2 + a_4\mu^4 + \&c.,$$

and by the series $\quad a_1\mu + a_3\mu^3 + a_5\mu^5 + \&c.,$

provided that in each series

$$a_{n+2} = -\frac{(i-s-n)(i+s+n+1)}{(n+1)(n+2)} a_n. \quad (57)$$

Hence, if we assume

$$\left.\begin{aligned} p_{is} & \text{ ast apr teristi } \frac{(i-s)(i+s+1)}{1.2}\mu^2 + \&c.\Big), \\ q_{is} &= a_1\Big(\mu \quad \frac{-s-1)(i+s+2)}{2.3}\mu^3 + \&c.\Big), \end{aligned}\right\} \quad (58)$$

$$Y_i = \Sigma\,(A_s \cos s\phi + B_s \sin s\phi)\,(1-\mu^2)^{\frac{s}{2}}(p_{is} + q_{is}), \quad (59)$$

where i and s have any values whatever, Y_i will be a spherical harmonic of the degree i.

If $i - s$ be a positive integer or $i + s$ a negative integer, one of the series p_{is} and q_{is} terminates, and the other contains an infinite number of terms.

In any other case, both series contain an infinite number of terms.

When the number of terms is infinite,

$$(1-\mu^2)^{\frac{s}{2}}p_{is} \quad \text{and} \quad (1-\mu^2)^{\frac{s}{2}}q_{is}$$

are each finite if $\mu < 1$; but if $\mu = 1$, each of these expressions becomes infinite.

In order to prove this, we observe that (57) may be written

$$a_{n+2} = \frac{n^2 + 3n + 2(s-1)n + (s-i)(s+i+1)}{n^2 + 3n + 2}\,a_n. \quad (60)$$

When n becomes very great, the ratio of a_{n+2} to a_n tends towards $1 + \dfrac{2(s-1)}{n}$.

Again, if we put $(1-\mu^2)^{-k} = 1 + b_2\mu^2 + b_4\mu^4 + \&c.$, we find that

$$b_{n+2} = \frac{n + 2 + 2\,(k-1)}{n+2}\,b_n.$$

As n becomes very great, the ratio of b_{n+2} to b_n tends towards

$$1 + \frac{2(k-1)}{n}.$$

Hence, as μ approaches 1, the functions p_{is} and q_{is} tend to become quantities of the same order as $(1-\mu^2)^{-s}$; and therefore, if s be positive,

$$(1-\mu^2)^{\frac{s}{2}}p_{is} \quad \text{and} \quad (1-\mu^2)^{\frac{s}{2}}q_{is}$$

are finite so long as $\mu < 1$; but if $\mu = 1$, they become infinite.

The same thing is true if s be negative. In this case, $(1-\mu^2)^{\frac{s}{2}}$ becomes infinite when $\mu = 1$, and the products

$$(1-\mu^2)^{\frac{s}{2}} p_{is} \quad \text{and} \quad (1-\mu^2)^{\frac{s}{2}} q_{is}$$

become infinite as before.

When $i = s$, we have $p_{is} = a_0$; and when $i = s+1$, the series $q_{is} = a_1 \mu$.

It appears from what has been said that if we assume

$$Y_i = (A \cos i\phi + B \sin i\phi)(1-\mu^2)^{\frac{i}{2}},$$

where i is positive, Y_i is always finite; but if i be not an integer, Y_i is not single-valued, for when ϕ increases by 2π the functions $\cos i\phi$ and $\sin i\phi$ do not return to their original values.

If $$V = \Sigma \frac{Y_i}{r^{i+1}}, \quad Y_i = A_i (1-\mu^2)^{\frac{i}{2}} \sin i\phi,$$

and $i = ns$, where n is any integer, and $s = \dfrac{\pi}{a}$, the function V satisfies Laplace's equation, vanishes at infinity, and is zero at the planes for which $\phi = 0$ and $\phi = a$. At the surface of a sphere of radius a we have

$$V = \Sigma \frac{Y_i}{a^{i+1}}.$$

Thus on this sphere V is a function of μ and ϕ, which vanishes at each of two great circles, and is finite and single-valued for the intercepted portion of the surface. By bringing in a sufficient number of terms, and properly determining the arbitrary constants, it may be possible to make this function equal, at least approximately, to an assigned function having the characteristics above.

In some cases, a function satisfying Laplace's equation, and fulfilling certain boundary conditions, can be found by

means of spherical harmonics of imaginary degrees. We have seen already that whatever be i and s,

$$\Sigma\,(A_s \cos s\phi + B_s \sin s\phi)\,(1 - \mu^2)^{\frac{s}{2}}(p_{is} + q_{is})$$

is a possible form of a spherical harmonic of the degree i. If i be imaginary in order that p_{is} and q_{is} should be real, it is necessary only that $i(i+1)$ should be real. If we put

$$i(i+1) = f, \quad \text{we get} \quad i = -\tfrac{1}{2} \pm \sqrt{(f + \tfrac{1}{4})}\,; \quad \text{whence,}$$

if i be imaginary, f must be negative and greater than $\frac{1}{4}$ in absolute magnitude. Accordingly, putting $f = -k$, we obtain

$$i = -\tfrac{1}{2} + \sqrt{-1}\,\sqrt{(k - \tfrac{1}{4})}, \quad i' = -\tfrac{1}{2} - \sqrt{-1}\,\sqrt{(k - \tfrac{1}{4})}\,;$$

then, since Y_i depends only on the value of $i(i+1)$, we have $Y_i = Y_{i'}$, and both these functions are real.

If we now assume

$$V = r^i Y_i + r^{i'} Y_{i'},$$

the function V satisfies Laplace's equation, and we have

$$V = (r^i + r^{i'})\,Y_i = \frac{Y_i}{\sqrt{r}}\,(e^{\chi\sqrt{-1}} + e^{-\chi\sqrt{-1}}) = \frac{2Y_i}{\sqrt{r}} \cos\chi.$$

If we assume $\sqrt{-1}\,V' = r^i Y_i - r^{i'} Y_{i'}$, we get in like manner $V' = \dfrac{2Y_i}{\sqrt{r}} \sin\chi$, where $\chi = \sqrt{(k - \tfrac{1}{4})}\,\log r$. In order that V or V' should vanish at a sphere of radius a, we have only to assume

$$\chi_a = \frac{2n+1}{2}\,\pi, \quad \text{or} \quad \chi_a = n\pi.$$

In the first case, we have

$$k = (n + \tfrac{1}{2})^2 \left(\frac{\pi}{\log a}\right)^2 + \tfrac{1}{4},$$

and, in the last,

$$k = \left(\frac{n\pi}{\log a}\right)^2 + \tfrac{1}{4}.$$

Incomplete spherical harmonics are here briefly described in order to give the student an idea of their nature and of the kind of conditions which they can be made to satisfy. They are useful in some departments of mathematical physics.

SECTION II.—*Ellipsoids of Revolution.*

154. **Solutions of Differential Equation.**—When the surfaces with which we have to do are not approximately spherical, the expansions for the potential which have been investigated are of little use. In the case of ellipsoids of revolution, equations (35) and (41), Art. 98, enable us, by an extension of the theory of spherical harmonics, to arrive at suitable forms for the potential.

Equation (41), Art. 98, if we write ϕ instead of χ, becomes by transposition

$$\frac{d}{d\xi}(1 - \xi^2)\frac{dV}{d\xi} + \frac{1}{1-\xi^2}\frac{d^2V}{d\phi^2} = \frac{d}{d\zeta}(1 - \zeta^2)\frac{dV}{d\zeta} + \frac{1}{1-\zeta^2}\frac{d^2V}{d\phi^2}. \tag{1}$$

If such a form be assigned to V as to make each member of this equation equal to the same quantity, the equation is satisfied; but, by Art. 148, if P_n satisfy equation (22), Art. 147, then

$$(1 - \mu^2)^{\frac{s}{2}} D^s P_n(A \cos s\phi + B \sin s\phi)$$

satisfies equation (27), Art. 148, if substituted for Y_n.

Hence

$$(1-\xi^2)^{\frac{s}{2}}(1-\zeta^2)^{\frac{s}{2}}\left(\frac{d}{d\xi}\right)^s\left(\frac{d}{d\zeta}\right)^s P_n(\xi)P_n(\zeta)(A\cos s\phi + B\sin s\phi)$$

must satisfy (1), and if V satisfy Laplace's equation throughout the region inside a prolate ellipsoid of revolution, we may put

$$V = \Sigma T_{ns}(\xi)T_{ns}(\zeta)\,(A_s\cos s\phi + B_s\sin s\phi). \qquad (2)$$

The value of V given by (2) becomes infinite along with ζ at points at an infinite distance from the centre of the ellipsoid. Accordingly, (2) does not give a suitable form for V in the space outside the ellipsoid.

It appears, however, from Art. 147 that there are two solutions of equation (22) of that Article. One of these is P_n; the other, which may be denoted by Q_n, contains only negative powers of μ. Accordingly, when ζ becomes infinite, $Q_n(\zeta)$ becomes zero.

Hence, if we put

$$\mathfrak{Y}_n=\Sigma(\xi^2-1)^{\frac{s}{2}}(\zeta^2-1)^{\frac{s}{2}}\left(\frac{d}{d\xi}\right)^s\left(\frac{d}{d\zeta}\right)^s P_n(\xi)(Q)_n(\zeta)(A_s\cos s\phi+B_s\sin s\phi), \qquad (3)$$

and $V = \Sigma\mathfrak{Y}_n$, (4), we see that this form of V satisfies Laplace's equation throughout the space outside the ellipsoid and is zero at infinity.

If we denote by U_{ns} the function corresponding to T_{ns} in equation (33), Art. 148, we have

$$U_{ns}(\zeta) = (\zeta^2-1)^{\frac{s}{2}}\left(\frac{d}{d\zeta}\right)^s Q_n(\zeta) \qquad (5)$$

and

$$\mathfrak{Y}_n = \Sigma T_{ns}(\xi)U_{ns}(\zeta)\,(A_s\cos s\phi + B_s\sin s\phi). \qquad (6)$$

At the surface of the ellipsoid, where ζ is constant, $\mathfrak{Y}_n$ becomes a spherical harmonic Y_n.

155. **Determination of the Function** Q_n. — The differential equation (22), Art. 147, being of the second order, has two particular integrals; one of these is P_n, the other Q_n. Putting ζ instead of μ, by Art. 147, we have

$$Q_n = K\left\{\zeta^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}\zeta^{-(n+3)} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}\zeta^{-(n+5)} + \&c.\right\} \tag{7}$$

By Art. 147, if a_s be the coefficient of ζ^s in this series,

$$a_{s-2} = \frac{s^2 - s}{s^2 - 3s - (n+2)(n-1)}\, a_s. \tag{8}$$

Hence, as $-s$ increases, the ratio $a_{s-2} : a_s$ tends to become unity, and if $\zeta > 1$, the series is convergent; but if $\zeta = 1$, it is divergent. Hence, in the space inside the ellipsoid of revolution, Q_n is not a suitable form for the potential.

156. **Determination of** Q_n **as an Integral.**—If we write y for S_n and x for μ, equation (22), Art. 147, becomes

$$\frac{d^2y}{dx^2} + \frac{2x}{x^2 - 1}\frac{dy}{dx} - \frac{n(n+1)}{x^2 - 1}y = 0, \tag{9}$$

which is of the form

$$\frac{d^2y}{dx^2} + X_1\frac{dy}{dx} + X_2 y = 0. \tag{10}$$

If we put $y = vy_1$, equation (10) becomes

$$v\left(\frac{d^2y_1}{dx^2} + X_1\frac{dy_1}{dx} + X_2 y_1\right) + y_1\frac{d^2v}{dx^2} + \left(2\frac{dy_1}{dx} + X_1 y_1\right)\frac{dv}{dx} = 0. \tag{11}$$

If y_1 be a solution of (10), we get

$$\frac{d\left(\dfrac{dv}{dx}\right)}{\dfrac{dv}{dx}} = -2\frac{dy_1}{y_1} - X_1 dx; \tag{12}$$

whence, by integration,

$$v = C_2 + C_1 \int \frac{e^{-\int X_1\,dx}}{y_1^2}\,dx,$$

and if $v = 0$ when $x = x_0$, we have

$$v = C_1 \int_{x_0}^{x} \frac{e^{-\int X_1\,dx}}{y_1^2}\,dx. \tag{13}$$

In the present case,

$$X_1 = \frac{2x}{x^2 - 1}, \quad \text{and} \quad e^{\int X_1\,dx} = x^2 - 1;$$

also, $y_1 = P_n$. Accordingly,

$$v = C_1 \int_{x_0}^{x} \frac{dx}{(x^2 - 1)P_n^2}. \tag{14}$$

By putting $x = \frac{1}{z}$, expanding the expression under the integral sign in ascending powers of z, and integrating, it is easy to see that when $x = \infty$, or $z = 0$, we have $v = 0$. Hence

$$Q_n = C_1 P_n \int_{\infty}^{x} \frac{dx}{(x^2 - 1)P_n^2}.$$

If we choose -1 for the value of C_1, we make Q_n perfectly definite, and we obtain

$$Q_n = P_n \int_{x}^{\infty} \frac{dx}{(x^2 - 1)P_n^2}. \tag{15}$$

157. **Expression of Q_n by means of a Finite Series.**—In order to express Q_n as a finite series, it is necessary first to prove some relations which exist between successive coefficients of Legendre and the functions obtained from them by differentiation.

If we put $x^2 - 1 = u$, $\dfrac{d}{dx} = D$, we have

$$Du = 2x, \quad D^2u = 2, \quad D^3u = 0;$$

and we get

$$\begin{aligned} D^{n+2}u^{n+1} &= D^{n+1}Du^{n+1} = D^{n+1}(n+1)u^n Du \\ &= (n+1)(Du\,D^{n+1}u^n + (n+1)D^2u\,D^nu^n) \\ &= 2(n+1)xD^{n+1}u^n + 2(n+1)^2D^nu^n. \end{aligned}$$

Substituting for $D^{n+1}u^n$ by a formula similar to that just obtained, we get

$$D^{n+2}u^{n+1} = 2(n+1)x\,\{2nxD^nu^{n-1} + 2n^2D^{n-1}u^{n-1}\} + 2(n+1)^2D^nu^n;$$

but by (21), Art. 147, we have

$$P_n = \frac{1}{2^n\,\underline{|n}}D^nu^n,$$

and therefore, dividing by $2^{n+1}\,\underline{|(n+1)}$, we get

$$DP_{n+1} = x^2DP_{n-1} + nxP_{n-1} + (n+1)P_n,$$

and subtracting DP_{n-1}, we have

$$DP_{n+1} - DP_{n-1} = uDP_{n-1} + nxP_{n-1} + (n+1)P_n. \qquad (16)$$

The right-hand side of (16) can be expressed in terms of P_n, for we have

$$D^nu^n = D^nu\,u^{n-1} = uD^nu^{n-1} + 2nxD^{n-1}u^{n-1} + \frac{2n(n-1)}{2}D^{n-2}u^{n-1};$$

also,

$$D^nu^n = D^{n-1}Du^n = D^{n-1}n\,u^{n-1}Du = 2nxD^{n-1}u^{n-1} + 2n(n-1)D^{n-2}u^{n-1}.$$

Comparing the two expressions for D^nu^n, we find

$$uD^nu^{n-1} = n(n-1)D^{n-2}u^{n-1}. \qquad (17)$$

Equation (17) shows that D^nu^n satisfies (22), Art. 147, a result which has been already proved in Art. 147. If we

now substitute $uD^n u^{n-1}$ for $n(n-1)D^{n-2}u^{n-1}$ in the first of the expressions for $D^n u^n$ given above, we get

$$D^n u^n = 2uD^n u^{n-1} + 2nxD^{n-1}u^{n-1}, \tag{18}$$

and dividing by $2^n \lfloor n$, we have

$$P_n = \frac{1}{n} uDP_{n-1} + xP_{n-1}, \tag{19}$$

whence $$uDP_{n-1} + nxP_{n-1} = nP_n.$$

Substituting in (16), we obtain

$$DP_{n+1} - DP_{n-1} = (2n+1)P_n. \tag{20}$$

From (20), we get immediately

$$DP_n = (2n-1)P_{n-1} + (2n-5)P_{n-3} \ldots + (2n-4s+3)P_{n-2s+1} + \&c. \tag{21}$$

We can now express Q_n as a finite series by treating the equation

$$D(uDy) - n(n+1)y = 0 \tag{22}$$

in a manner somewhat different from that previously employed. If we put $y = vy_1 - w$, and substitute in (22), we get

$$0 = \frac{d}{dx}(x^2-1)\frac{dvy_1}{dx} - n(n+1)vy_1 - \{D(uDw) - n(n+1)w\}$$

$$= \frac{d}{dx}\left\{(x^2-1)\left(v\frac{dy_1}{dx} + y_1\frac{dv}{dx}\right)\right\} - n(n+1)vy_1 - \{D(uDw) - n(n+1)w\}$$

$$= v\left\{\frac{d}{dx}(x^2-1)\frac{dy_1}{dx} - n(n+1)y_1\right\} + (x^2-1)\frac{dy_1}{dx}\frac{dv}{dx}$$

$$+ y_1\frac{d}{dx}(x^2-1)\frac{dv}{dx} + (x^2-1)\frac{dv}{dx}\frac{dy_1}{dx} - \{DuD - n(n+1)\}w.$$

If we next suppose y_1 to be a solution of (22), and determine v in such a manner as to satisfy

$$\frac{d}{dx}(1-x^2)\frac{dv}{dx} = 0, \tag{23}$$

we get
$$\{DuD - n(n+1)\}w = 2(x^2-1)\frac{dv}{dx}\frac{dy_1}{dx}. \tag{24}$$

From (23), by integration, we obtain

$$(1-x^2)\frac{dv}{dx} = \text{constant}.$$

If we choose 1 as the value of this constant, we get, by integration,

$$v = \tfrac{1}{2}\log\frac{x+1}{x-1}, \tag{25}$$

and (24) becomes

$$\frac{d}{dx}(1-x^2)\frac{dw}{dx} + n(n+1)w = 2\frac{dy_1}{dx}. \tag{26}$$

If we assume

$$w = A_1 P_{n-1} + A_3 P_{n-3} \ldots + A_{2s-1} P_{n-2s+1} + \&c.,$$

and make $y_1 = P_n$, by (21) of the present Article, and (2), Art. 138, we get

$$\begin{aligned}
&2(2n-4s+3)\\
&= A_{2s-1}\{n(n+1) - (n-2s+1)(n-2s+2)\}\\
&= A_{2s-1}\{n^2 + n - [n^2 - (4s-3)n + 2(s-1)(2s-1)]\}\\
&= A_{2s-1}\{(4s-2)n - 2(2s-1)(s-1)\} = 2(2s-1)(n-s+1)A_{2s-1},
\end{aligned}$$

whence
$$A_{2s-1} = \frac{2n-4s+3}{(2s-1)(n-s+1)}, \tag{27}$$

$$w = \frac{2n-1}{1.n}P_{n-1} + \frac{2n-5}{3(n-1)}P_{n-3} + \frac{2n-9}{5(n-2)}P_{n-5} + \&c. \tag{28}$$

Thus we obtain

$$Q_n = C\left\{P_n \tfrac{1}{2}\log\frac{x+1}{x-1} - w\right\}, \tag{29}$$

where w is given by (28).

We have now arrived at three expressions for Q_n of which one is perfectly definite, and the other two contain constants which can be determined by comparison with (15).

From (29) and (15) we get

$$C\left\{\tfrac{1}{2}P_n \log\frac{x+1}{x-1} - w\right\} = P_n \int_x^\infty \frac{dx}{(x^2-1)P_n^2},$$

whence, by division and differentiation, we get

$$C\left\{\frac{1}{1-x^2} - \frac{P_n\dfrac{dw}{dx} - w\dfrac{dP_n}{dx}}{P_n^2}\right\} = \frac{1}{(1-x^2)P_n^2}$$

that is,

$$C\left\{1 - \frac{1-x^2}{P_n^2}\left(P_n\frac{dw}{dx} - w\frac{dP_n}{dx}\right)\right\} = \frac{1}{P_n^2};$$

but when $x = 1$ we have $P_n = 1$, and therefore $C = 1$.

In order to find K in (7) we put $z = \dfrac{1}{x}$, and identify the coefficients of the lowest powers of z in the expressions for $Q_n(x)$ given by (7) and (15).

By Art. 147, we have

$$P_n = A\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \&c.\right\}$$

$= Ax^n\{1 + \text{ascending series in } z\}$, where $A = \dfrac{1\cdot3\cdot5\ldots(2n-1)}{\lfloor n}$;

therefore,

$$P_n\int_x^\infty \frac{dx}{(x^2-1)P_n^2} = Ax^n(1+\&c.)\int_0^z \frac{dz}{A^2z^2x^2(1-z^2)x^{2n}(1+\&c.)^2}$$

$$= \frac{x^n}{A}(1+\&c.)\int_0^z z^{2n}(1-z^2)^{-1}(1+\&c.)^{-2}\,dz.$$

Here the lowest power of z after integration is plainly z^{n+1}, and its coefficient is $\dfrac{1}{(2n+1)A}$; accordingly, $K = \dfrac{1}{(2n+1)A}$, that is,

$$K = \frac{|\underline{n}}{1.3.5\ldots(2n+1)}. \qquad (30)$$

For $Q(\zeta)$ we have, then, three expressions given by the equations

$$Q_n(\zeta) = \frac{|\underline{n}}{1.3.5\ldots(2n+1)}$$

$$\times\left\{\zeta^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}\zeta^{-(n+3)} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}\zeta^{-(n+5)} + \&c.\right\}, \qquad (31)$$

$$Q_n(\zeta) = P_n(\zeta)\int_\zeta^\infty \frac{d\zeta}{(\zeta^2-1)\{P_n(\zeta)\}^2}, \qquad (32)$$

$$Q_n(\zeta) = \tfrac{1}{2}P_n(\zeta)\log\frac{\zeta+1}{\zeta-1}$$

$$-\left\{\frac{2n-1}{1\,.\,n}P_{n-1} + \frac{2n-5}{3\,.\,(n-1)}P_{n-3} + \frac{2n-9}{5(n-2)}P_{n-5} + \&c.\right\}. \qquad (33)$$

158.—**Analogues of Tesseral Harmonics.**—We saw in Art. 148 that the multiplier of $u^{\frac{s}{2}}\cos s\phi$ in the spherical harmonic Y_n must satisfy equation (30) of that Article. This equation has two particular integrals, D^sP_n and D^sQ_n; but, by means of (30), Art. 148, the latter can be expressed in terms of the former. In fact, (30), Art. 148, may be written

$$(x^2-1)\frac{d^2y}{dx^2} + 2(s+1)x\frac{dy}{dx} + \{s(s+1) - n(n+1)\}y = 0. \qquad (34)$$

If we compare this with (10), Art. 156, we see that

$$X_1 = \frac{2(s+1)x}{x^2-1},$$

and consequently

$$e^{-\int X_1 dx} = (x^2-1)^{-(s+1)}.$$

Accordingly, if y_1 be a particular integral of (34), the other particular integral y_2 is given by the equation

$$y_2 = Cy_1 \int \frac{dx}{y_1^2 (x^2 - 1)^{s+1}};$$

and therefore, adopting the notation of Art. 154, if we put $T_{ns} = u^{\frac{s}{2}} y_1$, we have

$$U_{ns} = CT_{ns} \int_{x_0}^{x} \frac{dx}{y_1^2 (x^2 - 1)^{s+1}} = CT_{ns} \int_{x_0}^{x} \frac{dx}{(x^2 - 1)\, T_{ns}^2}. \qquad (35)$$

We may regard U_{ns} as defined by (5), Art. 154, and consider T_{ns} as given by the equation

$$T_{ns}(x) = (x^2 - 1)^{\frac{s}{2}} D^s P_n.$$

It is now easy to show that in (35) the value of C is

$$(-1)^{s+1} \frac{\lfloor n + s}{\lfloor n - s}.$$

For

$$T_{ns} = u^{\frac{s}{2}} D^s P_n = \frac{1.3.5\ldots(2n-1)}{\lfloor n} u^{\frac{s}{2}} D^s (x^n + \&c.)$$

$$= \frac{1.3.5\ldots(2n-1)\, n(n-1)\ldots(n-s+1)}{\lfloor n} u^{\frac{s}{2}} (x^{n-s} + \&c.)$$

$$= \frac{1.3.5\ldots(2n-1)}{\lfloor n - s} u^{\frac{s}{2}} (x^{n-s} + \&c.);$$

and if we put

$$\frac{1.3.5\ldots(2n-1)}{\lfloor n - s} = N, \quad \text{and} \quad \frac{1}{x} = z,$$

we get $T_{ns} = Nx^n (1 + Z)$, where Z denotes a series in

ascending powers of z, then

$$T\int_{x_0}^{x}\frac{dx}{(x^2-1)T^2}=Nx^n(1+Z)\int_{z}^{z_0}\frac{dz}{z^2x^2(1-z^2)N^2x^{2n}(1+Z)^2}$$

$$=\frac{x^n}{N}(1+Z)\int_{z}^{z_0}z^{2n}(1-z^2)^{-1}(1+Z)^{-2}dz=-\frac{z^{n+1}}{(2n+1)N}+\&c.; \tag{36}$$

but
$$Q_n=\frac{\lfloor n}{1.3.5\ldots(2n+1)}(x^{-(n+1)}+\&c.),$$

and

$$U_{ns}=u^{\frac{s}{2}}D^sQ_n=x^s(1-z^2)^{\frac{s}{2}}\frac{\lfloor n\,(n+1)(n+2)\ldots(n+s)}{1.3.5\ldots(2n+1)}(-1)^sz^{n+s+1}+\&c.$$

$$=(-1)^s\frac{\lfloor n+s}{1.3.5\ldots(2n+1)}z^{n+1}+\&c.$$

Hence, equating the coefficients of the lowest powers of z in the two expressions for U_{ns}, we get

$$-\frac{C}{(2n+1)N}=(-1)^s\frac{\lfloor n+s}{1.3.5\ldots(2n+1)},$$

whence

$$C=-(-1)^s\frac{\lfloor n+s}{\lfloor n-s};$$

and as $z_0=0$, and therefore $x_0=\infty$, we have

$$U_{ns}=(-1)^s\frac{\lfloor n+s}{\lfloor n-s}T_{ns}\int_{x}^{\infty}\frac{dx}{(x^2-1)T^2_{ns}}. \tag{37}$$

It is to be observed that, in order to avoid the introduction of imaginary quantities, $T_{ns}(x)$ is regarded as having a somewhat different signification according as $x<1$, or $x>1$.

In fact, in Art. 148,

$$T_{ns}(\mu) = (1-\mu^2)^{\frac{s}{2}} D^s P_n(\mu);$$

and, in the present Article,

$$T_{ns}(\xi) = (1-\xi^2)^{\frac{s}{2}} D^s P_n(\xi); \tag{38}$$

but

$$T_{ns}(\zeta) = (\zeta^2-1)^{\frac{s}{2}} D^s P_n(\zeta). \tag{39}$$

It is obvious that, whichever signification be attributed to $T_{ns}(x)$, it satisfies the same linear differential equation.

159. **Expansions for External and for Internal Potential.**—We can now write down the series expressing the potential, inside and outside a prolate ellipsoid of revolution, due to a distribution of mass on its surface.

Let V denote the potential inside, and V' that outside, the ellipsoid whose semi-axis major is $k\zeta_0$; then we may put

$$V = \Sigma\left\{A_n P_n(\xi) P_n(\zeta)\int_{\zeta_0}^{\infty}\frac{d\zeta}{(\zeta^2-1)\{P_n(\zeta)\}^2}\right.$$
$$\left.+\Sigma(A_{ns}\cos s\phi + B_{ns}\sin s\phi)T_{ns}(\xi)T_{ns}(\zeta)\int_{\zeta_0}^{\infty}\frac{d\zeta}{(\zeta^2-1)\{T_{ns}(\zeta)\}^2}\right\},$$

and (40)

$$V' = \Sigma\left\{A_n P_n(\xi) P_n(\zeta)\int_{\zeta}^{\infty}\frac{d\zeta}{(\zeta^2-1)\{P_n(\zeta)\}^2}\right.$$
$$\left.+\Sigma(A_{ns}\cos s\phi + B_{ns}\sin s\phi)T_{ns}(\xi)T_{ns}(\zeta)\int_{\zeta}^{\infty}\frac{d\zeta}{(\zeta^2-1)\{T_{ns}(\zeta)\}^2}\right\}. \tag{41}$$

At the surface, where $\zeta = \zeta_0$, these two expressions become the same series of spherical harmonics which can be made equal to any assigned function of ξ and ϕ which is finite and single-valued.

160. **Surface Distribution corresponding to Potential.**—If the internal and external potentials, V and V', due to a surface distribution whose density is σ, be given by the equations

$$V = A\cos s\phi\, T(\xi)\, T(\zeta)\int_{\zeta_0}^{\infty}\frac{d\zeta}{(\zeta^2-1)\{T(\zeta)\}^2},$$

$$V' = A\cos s\phi\, T(\xi)\, T(\zeta)\int_{\zeta}^{\infty}\frac{d\zeta}{(\zeta^2-1)\{T(\zeta)\}^2},$$

σ can be found from the equation at the surface of the ellipsoid ζ_0.

If ds_1 be an element, drawn outwards, of the normal to the surface, equation (12), Art. 46, becomes

$$\frac{dV}{ds_1} - \frac{dV'}{ds_1} = 4\pi\sigma;$$

but by Ex. 2, Art. 75, $ds_1 = \dfrac{k^2\zeta d\zeta}{p}$, and therefore we have

$$\frac{4\pi k^2\zeta}{p}\sigma = \frac{dV}{d\zeta} - \frac{dV'}{d\zeta} = A\cos s\phi\, T(\xi)$$

$$\times\left\{\frac{dT}{d\zeta}\int_{\zeta_0}^{\infty}\frac{d\zeta}{(\zeta^2-1)T^2} - \frac{dT}{d\zeta}\int_{\zeta_0}^{\infty}\frac{d\zeta}{(\zeta^2-1)T^2} + T(\zeta_0)\frac{1}{(\zeta_0^2-1)\{T(\zeta_0)\}^2}\right\}$$

$$= \frac{A}{(\zeta_0^2-1)T(\zeta_0)}T(\xi)\cos s\phi; \quad \text{whence}$$

$$4\pi\sigma = \frac{Ap}{k^2\zeta_0(\zeta_0^2-1)T(\zeta_0)}T(\xi)\cos s\phi = \frac{A\,T(\xi)\cos s\phi}{kT(\zeta_0)\{(\zeta_0^2-1)(\zeta_0^2-\xi^2)\}^{\frac{1}{2}}}. \tag{42}$$

When the density of the surface distribution is given, the expressions for the potential inside and outside the ellipsoid may be determined by expanding $\dfrac{\sigma}{p}$, expressed as a function of ξ and ϕ in a series of spherical harmonics, and determining each of the functions $T(\xi)$ and the corresponding constants by means of (42). The potentials V and V' are then given by (40) and (41).

161. Potentials of Homœoid and Focaloid.—If the surface distribution be homœoidal, the density varies as p, and the multiplier of p on the right-hand side of (42) must be constant. Hence, V = constant,

$$V' = C\int_\zeta^\infty \frac{d\zeta}{\zeta^2 - 1} = \tfrac{1}{2}C\log\frac{\zeta+1}{\zeta-1} = C\left\{\frac{1}{\zeta} + \frac{1}{3}\,\frac{1}{\zeta^3} + \text{\&c.}\right\}. \tag{43}$$

When ζ becomes very great, V' tends to become $\dfrac{M}{r}$, that is $\dfrac{M}{k\zeta}$, where M denotes the total mass of the homœoid.

Hence $C = \dfrac{M}{k}$, and

$$V' = \frac{M}{2k}\log\frac{\zeta+1}{\zeta-1}, \quad V = \frac{M}{2k}\log\left(\frac{\zeta_0+1}{\zeta_0-1}\right). \tag{44}$$

For a focaloidal distribution the density varies inversely as p, and $\dfrac{\sigma}{p}$ varies as $\dfrac{1}{p^2}$, that is, as $\zeta^2{}_0 - \xi^2$. Accordingly, $\dfrac{\sigma}{p}$ is of the form $AP_0 + BP_2(\xi)$, and

$$V' = A_0\int_\zeta^\infty \frac{d\zeta}{\zeta^2-1} + A_2(\xi^2 - \tfrac{1}{3})(\zeta^2 - \tfrac{1}{3})\int_\zeta^\infty \frac{d\zeta}{(\zeta^2-1)(\zeta^2-\frac{1}{3})^2}.$$

It is easily seen that

$$\frac{1}{\zeta^2-1} = \frac{1}{2}\left(\frac{1}{\zeta-1} - \frac{1}{\zeta+1}\right),$$

and that

$$\frac{1}{(\zeta^2-1)(\zeta^2-\frac{1}{3})^2} = \frac{9}{8}\left\{\frac{1}{\zeta-1} - \frac{1}{\zeta+1} - \frac{1}{\left(\zeta+\frac{1}{\sqrt{3}}\right)^2} - \frac{1}{\left(\zeta-\frac{1}{\sqrt{3}}\right)^2}\right\}.$$

Hence, by integration, we have

$$V' = \frac{A_0}{2}\log\frac{\zeta+1}{\zeta-1} + \frac{9}{8}A_2(\xi^2-\tfrac{1}{3})\left\{(\zeta^2-\tfrac{1}{3})\log\frac{\zeta+1}{\zeta-1} - 2\zeta\right\}. \tag{45}$$

By a method similar to that employed in the case of the homœoid, we find that $A_0 = \frac{M}{k}$, where M is the mass of the focaloid.

It follows from (42) that the coefficients of P_0 and $(\xi^2 - \frac{1}{3})$ in $4\pi k^2\zeta_0(\zeta_0^2 - 1)\frac{\sigma}{p}$ are A_0 and $\frac{A_2}{\zeta_0^2 - \frac{1}{3}}$; but $\frac{\sigma}{p}$ varies as $\zeta_0^2 - \xi^2$, that is, as $\zeta_0^2 - \frac{1}{3} - (\xi^2 - \frac{1}{3})$, and therefore we must have $A_2 = -A_0$. Hence

$$V' = \frac{M}{2k}\left\{\log\left(\frac{\zeta+1}{\zeta-1}\right) - \frac{9}{4}\left(\xi^2 - \frac{1}{3}\right)\left[\left(\zeta^2 - \frac{1}{3}\right)\log\left(\frac{\zeta+1}{\zeta-1}\right) - 2\zeta\right]\right\}. \tag{46}$$

By Art. 83, the potential of a focaloid in external space is the same as that of the solid ellipsoid of equal mass of which it is the boundary. Hence (46) may be verified by comparing it with (2), Art. 78. This verification is readily effected by taking a point on the axis of revolution. Here $\xi = 1$, and $r = k\zeta$; then putting $\frac{1}{\zeta} = z$, from (46) we get

$$V' = \frac{M}{k}\left\{z + \frac{z^3}{3} + \&c. - \frac{3}{2}\left[\left(\zeta^2 - \frac{1}{3}\right)\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \&c.\right) - \zeta\right]\right\}$$

$$= \frac{M}{k}\left\{z + z^3\left[\frac{1}{3} - \frac{3}{2}\left(\frac{1}{5} - \frac{1}{9}\right)\right] + \&c.\right\} = \frac{M}{k}\left(z + \frac{z^3}{5} + \&c.\right). \tag{47}$$

Again in (2), Art. 78, for points on the axis of revolution, $I = A$; and, since $C = B$, we have

$$\frac{A + B + C - 3I}{2} = B - A = \frac{Mk^2}{5}.$$

Hence, putting $k\zeta$ for r, we get

$$V = \frac{M}{k\zeta} + \frac{M}{5k\zeta^3},$$

which agrees with (47).

162. **Oblate Ellipsoid of Revolution.**—When we have to do with oblate ellipsoids of revolution, Laplace's equation takes the form given by (35), Art. 98. If we put $\zeta = -\zeta'\sqrt{-1}$ in this equation, and write ϕ instead of χ, we get an equation in ζ', ξ, and ϕ, which is the same as (1). Hence, in the case of the oblate ellipsoid, we may put for V and V' the expressions given by (2) and (4), or by (40) and (41), provided we put ζ' instead of ζ. In order to determine completely these expressions for V and V', we have then to put $\zeta\sqrt{-1}$ for ζ', and accordingly we have to find what $P(\zeta')$, $Q(\zeta')$, $T(\zeta')$, and $U(\zeta')$ become when $\zeta\sqrt{-1}$ is substituted for ζ'.

163. **Determination of $p(\zeta)$ and $q(\zeta)$.** If we put $\zeta\sqrt{-1}$ instead of ζ in the expression for $P_n(\zeta)$, given by (26), Art. 147, we get

$$(-1)^{\frac{n}{2}}A\left\{\zeta^n + \frac{n(n-1)}{2(2n-1)}\zeta^{n-2} + \&c.\right\},$$

and we may put

$$p_n(\zeta) = \frac{1.3.5\ldots(2n-1)}{\underline{|n}}\left\{\zeta^n + \frac{n(n-1)}{2(2n-1)}\zeta^{n-2} + \&c.\right\}. \tag{48}$$

Also, putting $\sqrt{-1} = \iota$, we have

$$P_n(\iota\zeta) = \iota^n p_n(\zeta). \tag{49}$$

In like manner, from (31) and (32), we have

$$q_n(\zeta) = \frac{\underline{|n}}{1.3.5 - (2n+1)}\left\{\zeta^{-(n+1)} - \frac{(n+1)(n+2)}{2.(2n+3)}\zeta^{-(n+3)} + \&c.\right\}, \tag{50}$$

$$q_n(\zeta) = p_n(\zeta)\int_\zeta^\infty \frac{d\zeta}{(\zeta^2+1)\{p_n(\zeta)\}^2}. \tag{51}$$

It is plain that the right-hand members of (50) and (51) can differ only by a factor which is some power of ι from the expressions for $Q(\iota\zeta)$, given by (31) and (32); and, as the term involving the lowest power of $\frac{1}{\zeta}$ is the same in the two forms of $q_n(\zeta)$, they are consistent.

From (31), we see that

$$Q_n(\iota\zeta) = \iota^{-(n+1)} q_n(\zeta). \tag{52}$$

In order to find a third expression for $q_n(\zeta)$, we must consider what $\log \frac{\zeta+1}{\zeta-1}$ becomes when ζ is changed into $\iota\zeta$.

If we put $\frac{1}{\zeta} = z = \tan\theta$, we have

$$\log \frac{\iota\zeta+1}{\iota\zeta-1} = \log \frac{1-\iota z}{1+\iota z} = \log \frac{\cos\theta - \iota\sin\theta}{\cos\theta + \iota\sin\theta} = -2\iota\theta.$$

Hence,

$$\tfrac{1}{2}\log \frac{\iota\zeta+1}{\iota\zeta-1} = \iota^3 \tan^{-1} z,$$

and

$$\tfrac{1}{2} P_n(\iota\zeta) \log \frac{\iota\zeta+1}{\iota\zeta-1} = \iota^{n+3} p_n(\zeta) \tan^{-1}\frac{1}{\zeta};$$

also,

$$P_{n-1}(\iota\zeta) = \iota^{n-1} p_{n-1}(\zeta).$$

Accordingly, by (33),

$$Q_n(\iota\zeta) = \iota^{n+3}\left\{p_n(\zeta)\tan^{-1}\frac{1}{\zeta} - \left(\frac{2n-1}{1.n}p_{n-1} - \frac{2n-5}{3(n-1)}p_{n-3} + \&c.\right)\right\};$$

but $q_n(\zeta) = \iota^{n+1} Q_n(\iota\zeta)$, and $\iota^{2n+4} = (-1)^n$; and therefore

$$q_n(\zeta) = (-1)^n\left\{p_n(\zeta)\tan^{-1}\frac{1}{\zeta} - \frac{2n-1}{1.n}p_{n-1}(\zeta) + \frac{2n-5}{3(n-1)}p_{n-3}(\zeta) - \&c.\right\}$$

164. **Analogues of Tesseral Harmonics.**—When we put $\iota\zeta$ for ζ in $T_{ns}(\zeta)$, we get

$$\iota^n(\zeta^2 + 1)^{\frac{s}{2}} D^s p_n(\zeta),$$

and we may write

$$t_{ns}(\zeta) = (\zeta^2 + 1)^{\frac{s}{2}} D^s p_n(\zeta), \quad T_{ns}(\iota\zeta) = \iota^n t_{ns}(\zeta). \quad (54)$$

Also, we may write

$$u_{ns}(\zeta) = (\zeta^2 + 1)^{\frac{s}{2}} D^s q_n(\zeta)\,; \quad (55)$$

whence

$$U_{ns}(\iota\zeta) = \iota^{-(n+1)} u_{ns}(\zeta). \quad (56)$$

Another form of $u_{ns}(\zeta)$ is obtained from (37) from which, by means of (56), we have

$$u_{ns}(\zeta) = \iota^{n+1} U_{ns}(\iota\zeta) = \iota^{n+1}(-1)^s \times \frac{\lfloor n+s}{\lfloor n-s} \iota^n t_{ns}(\zeta) \int_\zeta^\infty \frac{-\iota d\zeta}{\iota^{2n}(\zeta^2+1)\{t_{ns}(\zeta)\}^2};$$

whence

$$u_{ns}(\zeta) = (-1)^s \frac{\lfloor n+s}{\lfloor n-s} t_{ns}(\zeta) \int_\zeta^\infty \frac{d\zeta}{(\zeta^2+1)\{t_{ns}(\zeta)\}^2}. \quad (57)$$

165. **Expression for Potentials.**—If V and V' denote the potentials inside and outside an oblate ellipsoid of revolution, due to a distribution of mass on its surface, we may write

$$V = \Sigma A_n P_n(\xi) p_n(\zeta) \int_{\zeta_0}^\infty \frac{d\zeta}{(\zeta^2+1)\{p_n(\zeta)\}^2} + \Sigma(A_{ns}\cos s\phi + B_{ns}\sin s\phi)\, T_{ns}(\xi) t_{ns}(\zeta) \int_{\zeta_0}^\infty \frac{d\zeta}{(\zeta^2+1)\{t_{ns}(\zeta)\}^2}, \quad (58)$$

$$V' = \Sigma A_n P_n(\xi) p_n(\zeta) \int_\zeta^\infty \frac{d\zeta}{(\zeta^2+1)\{p_n(\zeta)\}^2} + \Sigma(A_{ns}\cos s\phi + B_{ns}\sin s\phi)\, T_{ns}(\xi) t_{ns}(\zeta) \int_\zeta^\infty \frac{d\zeta}{(\zeta^2+1)\{t_{ns}(\zeta)\}^2}. \quad (59)$$

It is plain from what precedes that V and V' satisfy each Laplace's equation, that V is finite when $\zeta = 0$, and V' is zero when $\zeta = \infty$, and that V and V' are identical at the surface of the ellipsoid. Hence they satisfy all the conditions required.

166. **Surface Distribution corresponding to Potential.**—Here we may proceed as in Art. 160. If the internal and external potentials V and V' be given by the equations

$$\left.\begin{aligned} V &= A \cos s\phi\, T(\xi)\, t(\zeta) \int_{\zeta_0}^{\infty} \frac{d\zeta}{(\zeta^2+1)\{t(\zeta)\}^2} \\ V' &= A \cos s\phi\, T(\xi)\, t(\zeta) \int_{\zeta}^{\infty} \frac{d\zeta}{(\zeta^2+1)\{t(\zeta)\}^2} \end{aligned}\right\}, \qquad (60)$$

and ds_1 be an element of the normal to the ellipsoid,

$$ds_1 = \frac{\lambda d\lambda}{p};$$

but in this case, by Art. 98, we have $\lambda^2 = k^2(\zeta^2+1)$, and therefore

$$ds_1 = \frac{k^2\zeta d\zeta}{p}.$$

Accordingly, as in Art. 160, we get

$$\frac{4\pi k^2 \zeta_0}{p}\sigma = \frac{A}{(\zeta_0^2+1)\,t(\zeta_0)}\, T(\xi) \cos s\phi,$$

and

$$4\pi\sigma = \frac{Ap}{k^2\zeta_0(\zeta_0^2+1)\,t(\zeta_0)}\, T(\xi)\cos s\phi = \frac{A\,T(\xi)\cos s\phi}{k\,t(\zeta_0)\sqrt{\{(\zeta_0^2+1)(\zeta_0^2+\xi^2)\}}}. \qquad (61)$$

When the density of the surface distribution is given, the potentials may be determined in a manner similar to that described in Art. 160.

167. **Potentials of Homœoid and Focaloid.**—These may be obtained from (60) and (61) in a manner similar to that employed in Art. 161 ; but if the expressions given by (44) and (46) are already known, we can get from them the corresponding expressions for an oblate ellipsoid.

Putting $\iota\zeta$ for ζ in (44), we get

$$\iota^3 \frac{M}{k} \tan^{-1} \frac{1}{\zeta}.$$

Hence we may conclude that V', the potential of an oblate homœoid of revolution, is given by the equation

$$V' = \frac{M}{k} \tan^{-1} \frac{1}{\zeta}. \qquad (62)$$

It is easy to see that this expression for V' must be correct, since it satisfies Laplace's equation, vanishes at ∞, is constant at the surface, and tends towards the value $\frac{M}{k\zeta}$ at points very distant from the centre when r tends to become equal to $k\zeta$.

To get the potential of a focaloid in external space, we may put $\iota\zeta$ for ζ in (46), and we get

$$\frac{M}{k}\left\{\iota^3 \tan^{-1}\frac{1}{\zeta} - \frac{9}{4}\left(\xi^2 - \frac{1}{3}\right)\left[\iota^5\left(\zeta^2 + \frac{1}{3}\right)\tan^{-1}\frac{1}{\zeta} - \iota\zeta\right]\right\};$$

since $\iota = \iota^5$, and $\iota^2 = -1$, this may be written

$$\iota^3 \frac{M}{k}\left\{\tan^{-1}\frac{1}{\zeta} + \frac{9}{4}\left(\xi^2 - \frac{1}{3}\right)\left[\left(\zeta^2 + \frac{1}{3}\right)\tan^{-1}\frac{1}{\zeta} - \zeta\right]\right\}.$$

Hence we may put

$$V' = \frac{M}{k}\left\{\tan^{-1}\frac{1}{\zeta} + \frac{9}{4}\left(\xi^2 - \frac{1}{3}\right)\left[\left(\zeta^2 + \frac{1}{3}\right)\tan^{-1}\frac{1}{\zeta} - \zeta\right]\right\}. \qquad (63)$$

This expression for V' satisfies Laplace's equation, vanishes at infinity, and at points very distant from the centre tends towards the value $\frac{M}{r}$; but to prove that it satisfies all the conditions of the question, we must show that the corresponding distribution of mass varies as $\frac{1}{p}$.

If the external potential V' be given by (63), the internal potential V, due to the surface-distribution producing V', is given by the equation

$$V = \frac{M}{k}\left\{\tan^{-1}\frac{1}{\zeta_0} + \frac{9}{4}\left(\xi^2 - \frac{1}{3}\right)\left(\zeta^2 + \frac{1}{3}\right)\left[\tan^{-1}\frac{1}{\zeta_0} - \frac{\zeta_0}{\zeta_0^2 + \frac{1}{3}}\right]\right\}, \tag{64}$$

since this expression for V satisfies Laplace's equation, remains finite inside the ellipsoid, and is equal to V' at the surface. We have, then,

$$\frac{4\pi k^2\zeta_0}{p}\sigma = \frac{dV}{d\zeta} - \frac{dV'}{d\zeta}$$

$$= \frac{M}{k}\left\{\frac{1}{\zeta^2\left(1 + \frac{1}{\zeta^2}\right)} - \frac{9}{4}\left(\xi^2 - \frac{1}{3}\right)\left(\zeta^2 + \frac{1}{3}\right)\frac{d}{d\zeta}\left(\tan^{-1}\frac{1}{\zeta} - \frac{\zeta}{\zeta^2 + \frac{1}{3}}\right)\right\} \tag{65}$$

Differentiating and reducing, it is easy to see that

$$\left(\zeta^2 + \frac{1}{3}\right)\frac{d}{d\zeta}\left(\tan^{-1}\frac{1}{\zeta} - \frac{\zeta}{\zeta^2 + \frac{1}{3}}\right) = \frac{\zeta^2 - \frac{1}{3}}{\zeta^2 + \frac{1}{3}} - \frac{\zeta^2 + \frac{1}{3}}{1 + \zeta^2} = \frac{-\frac{4}{9}}{(\zeta^2 + \frac{1}{3})(1 + \zeta^2)}.$$

Hence,

$$\frac{dV}{d\zeta} - \frac{dV'}{d\zeta} = \frac{M}{k}\left\{\frac{1}{1 + \zeta^2} + \frac{\xi^2 - \frac{1}{3}}{(1 + \zeta^2)(\zeta^2 + \frac{1}{3})}\right\} = \frac{M}{k}\,\frac{\zeta^2 + \xi^2}{(1 + \zeta^2)(\zeta^2 + \frac{1}{3})},$$

and at the surface we have

$$\frac{4\pi k^2\zeta_0}{p}\sigma = \frac{dV}{d\zeta} - \frac{dV'}{d\zeta} = \frac{M}{k}\frac{\zeta_0^2 + \xi^2}{(1+\zeta_0^2)(\zeta_0^2+\frac{1}{3})} = \frac{C}{p^2},$$

where C denotes a constant.

Accordingly $\sigma = \dfrac{C}{4\pi k\,\zeta_0}\dfrac{1}{p}$, and therefore the distribution of mass producing the potential is focaloidal.

Section III.—*Ellipsoids in General.*

168. **Ellipsoidal Harmonics.**—When the surface, at which the potential or mass-distribution is given, is an ellipsoid not of revolution, the preceding methods are inapplicable. The most general method of determining solutions of Laplace's equation which can be made use of in questions of this kind depends on the employment of functions called ellipsoidal harmonics.

We have seen, Art. 92, that if λ, μ, ν be the primary semi-axes of the three confocal quadrics passing through a point, Laplace's equation may be written in the form

$$(\mu^2 - \nu^2)\frac{d^2V}{d\alpha^2} + (\lambda^2 - \nu^2)\frac{d^2V}{d\beta^2} + (\lambda^2 - \mu^2)\frac{d^2V}{d\gamma^2} = 0, \quad (1)$$

where α, β, γ are given by (17), Art. 92. If a, b, c denote the semi-axes of an ellipsoid of the confocal system, we may change the variables by assuming

$$\lambda^2 = a^2 + \xi, \quad \mu^2 = a^2 + \eta, \quad \nu^2 = a^2 + \zeta.$$

If we put $\mathfrak{X} = (a^2+\xi)^{\frac{1}{2}}(b^2+\xi)^{\frac{1}{2}}(c^2+\xi)^{\frac{1}{2}}$, by (17), Art. 92, we have

$$\frac{d}{d\alpha} = \frac{d\lambda}{d\alpha}\frac{d\xi}{d\lambda}\frac{d}{d\xi} = \frac{2\mathfrak{X}}{k}\frac{d}{d\xi}. \quad (2)$$

If we assume

$$\mathfrak{Y} = (a^2+\eta)^{\frac{1}{2}}(b^2+\eta)^{\frac{1}{2}}(c^2+\eta)^{\frac{1}{2}}, \quad \mathfrak{Z} = (a^2+\zeta)^{\frac{1}{2}}(b^2+\zeta)^{\frac{1}{2}}(c^2+\zeta)^{\frac{1}{2}},$$

in like manner we get

$$\frac{d}{d\beta} = \frac{2\mathfrak{Y}\sqrt{-1}}{k}\frac{d}{d\eta}, \quad \frac{d}{d\gamma} = \frac{2\mathfrak{Z}}{k}\frac{d}{d\zeta}; \tag{3}$$

and Laplace's equation becomes

$$\left\{(\eta-\zeta)\left(\mathfrak{X}\frac{d}{d\xi}\right)^2 + (\zeta-\xi)\left(\mathfrak{Y}\frac{d}{d\eta}\right)^2 + (\xi-\eta)\left(\mathfrak{Z}\frac{d}{d\zeta}\right)^2\right\} V = 0. \tag{4}$$

Following the analogy suggested by the methods employed in the case of ellipsoids of revolution, we may suppose V to be the product of E, a function of ξ, and of H a function of η and H' a function of ζ. If these functions be such that

$$\left(\mathfrak{X}\frac{d}{d\xi}\right)^2 E = (m\xi+j)E, \quad \left(\mathfrak{Y}\frac{d}{d\eta}\right)^2 \mathrm{H} = (m\eta+j)\mathrm{H},$$

and

$$\left(\mathfrak{Z}\frac{d}{d\zeta}\right)^2 \mathrm{H}' = (m\zeta+j)H', \tag{5}$$

where m and j are disposable constants; we may put $V = CEHH'$, where C is an arbitrary constant, and we have

$$\nabla^2 V = \Lambda\{(\eta-\zeta)(m\xi+j)+(\zeta-\xi)(m\eta+j)+(\xi-\eta)(m\zeta+j)\} V = 0,$$

where Λ denotes the other factor of $\nabla^2 V$, since the expression inside the bracket vanishes identically.

We have now to find forms of the functions E, H, and H' which will satisfy (5).

If we suppose E to be a rational and integral function of ξ, we may put

$$E = \xi^n + p_1\xi^{n-1} + \ldots + p_n.$$

Operating on ξ^n, we get

$$\left(\mathfrak{X}\frac{d}{d\xi}\right)^2 \xi^n = n(n-1)\,\mathfrak{X}^2\xi^{n-2}$$
$$+\frac{n\{(a^2+\xi)(b^2+\xi)+(b^2+\xi)(c^2+\xi)+(c^2+\xi)(a^2+\xi)\}}{2}\,\xi^{n-1}$$

$$= n(n+\tfrac{1}{2})\xi^{n+1} + n^2(a^2+b^2+c^2)\xi^n + n(n-\tfrac{1}{2})(a^2b^2+b^2c^2+c^2a^2)\xi^{n-1}$$
$$+ n(n-1)\,a^2b^2c^2\xi^{n-2}.$$

Hence, in order that

$$\left(\mathfrak{X}\frac{d}{d\xi}\right)^2 E = (m\xi + j)E,$$

we must have the series of $n+2$ equations

$$n(n+\tfrac{1}{2}) = m, \qquad (n-1)(n-\tfrac{1}{2})p_1 + n^2(a^2+b^2+c^2) = j + mp_1,$$

$$n(n-\tfrac{1}{2})(a^2b^2+b^2c^2+c^2a^2) + (n-1)^2(a^2+b^2+c^2)p_1$$
$$+ (n-2)(n-\tfrac{3}{2})p_2 = jp_1 + mp_2, \quad \&c.,$$

$$2a^2b^2c^2p_{n-2} + \tfrac{1}{2}(a^2b^2+b^2c^2+c^2a^2)p_{n-1} = jp_n.$$

The first of these determines m, the second p_1 as a linear function of j. By substituting this value of p_1 in the third, p_2 is determined as a quadratic function of j, and so on. Thus, finally, an equation of the degree $n+1$ is obtained to determine j. Each root of this equation corresponds to a set of values of p_1, p_2, &c., which furnishes a function of the required form for E. There are thus $n+1$ functions of the degree n in ξ which are of the required form. It is plain that ξ is of the second degree in the coordinates x, y, z. Hence the forms of V corresponding to those found for E must be of even degree in each of the coordinates x, y, z. To determine forms of odd degree in these coordinates, we are guided by the formulæ for expressing the Cartesian in terms of the elliptic coordinates of a point.

In fact (Salmon's *Geometry of Three Dimensions*),

$$x^2 = \frac{(a^2+\xi)(a^2+\eta)(a^2+\zeta)}{(a^2-b^2)(a^2-c^2)}, \quad y^2 = \frac{(b^2+\xi)(b^2+\eta)(b^2+\zeta)}{(b^2-a^2)(b^2-c^2)},$$

$$z^2 = \frac{(c^2+\xi)(c^2+\eta)(c^2+\zeta)}{(c^2-a^2)(c^2-b^2)}. \qquad (6)$$

If we consider only the factor containing ξ, we see that x corresponds to

$$\sqrt{a^2+\xi}, \quad y \text{ to } \sqrt{b^2+\xi}, \quad \text{and} \quad z \text{ to } \sqrt{c^2+\xi};$$

and we are thus led to consider whether

$$\sqrt{a^2+\xi}\,\Xi, \quad \sqrt{(a^2+\xi)(b^2+\xi)}\,\Xi,$$

and

$$\sqrt{(a^2+\xi)(b^2+\xi)(c^2+\xi)}\,\Xi,$$

where Ξ is a rational and integral function of ξ, are possible forms of E.

Operating on $\sqrt{a^2+\xi}\,\xi^n$, we get

$$\left.\begin{aligned}&\left(\mathfrak{X}\frac{d}{d\xi}\right)^2\sqrt{(a^2+\xi)}\,\xi^n = \sqrt{(a^2+\xi)}\,\{(n+1)(n+\tfrac{1}{2})\xi^{n+1}\\ &+[n^2a^2+(n+\tfrac{1}{2})^2(b^2+c^2)]\xi^n+[n(n-\tfrac{1}{2})a^2(b^2+c^2)\\ &+n(n+\tfrac{1}{2})b^2c^2]\xi^{n-1}+n(n-1)a^2b^2c^2\xi^{n-2}\}\end{aligned}\right\}. \qquad (7)$$

Hence, we see that if $E = \sqrt{a^2+\xi}\,\Xi_n$, where Ξ_n is a rational and integral function of ξ of the degree n, we have

$$\left(\mathfrak{X}\frac{d}{dx}\right)^2 E = \sqrt{a^2+\xi}\,\Xi_{n+1},$$

and therefore, that by properly determining m, j, and the coefficients in Ξ_n, we can make

$$\left(\mathfrak{X}\frac{d}{d\xi}\right)^2 E = (m\xi + j)E.$$

It is easy to see that the final equation for determining j is of the degree $n+1$.

If we operate on the functions

$$\sqrt{\{(a^2+\xi)(b^2+\xi)\}}\,\xi^n, \quad \text{and} \quad \sqrt{\{(a^2+\xi)(b^2+\xi)(c^2+\xi)\}}\,\xi^n,$$

we find that

$$\left.\begin{aligned}\left(\mathfrak{X}\frac{d}{d\xi}\right)^2 \sqrt{\{(a^2+\xi)(b^2+\xi)\}}\,\xi^n &= \sqrt{\{(a^2+\xi)(b^2+\xi)\}}\,\{(n+1)(n+\tfrac{3}{2})\xi^{n+1} \\ &+ [(n+\tfrac{1}{2})^2(a^2+b^2)+(n+1)^2c^2]\,\xi^n + [n(n-\tfrac{1}{2})\,a^2b^2 \\ &+ n(n+\tfrac{1}{2})(a^2+b^2)c^2]\,\xi^{n-1} + n\,(n-1)\,a^2b^2c^2\,\xi^{n-2}\}\end{aligned}\right\}, \tag{8}$$

and that

$$\left.\begin{aligned}&\left(\mathfrak{X}\frac{d}{d\xi}\right)^2 \sqrt{\{(a^2+\xi)(b^2+\xi)(c^2+\xi)\}}\,\xi^n \\ &= \mathfrak{X}\{(n+\tfrac{3}{2})\,(n+2)\,\xi^{n+1} + (n+1)^2(a^2+b^2+c^2)\,\xi^n + n \\ &\times (n+\tfrac{1}{2})\,(a^2b^2+b^2c^2+c^2a^2)\,\xi^{n-1} + n\,(n-1)\,a^2b^2c^2\,\xi^{n-2}\}\end{aligned}\right\}. \tag{9}$$

Hence we conclude that

$$\left(\mathfrak{X}\frac{d}{d\xi}\right)^2 \sqrt{(a^2+\xi)(b^2+\xi)}\,\Xi_n = \sqrt{(a^2+\xi)(b^2+\xi)}\,\Xi_{n+1},$$

and that
$$\left(\mathfrak{X}\frac{d}{d\xi}\right)^2 \mathfrak{X}\,\Xi_n = \mathfrak{X}\,\Xi_{n+1},$$

and, accordingly, that $\sqrt{\{(a^2+\xi)(b^2+\xi)\}}\,\Xi_n$ and $\mathfrak{X}\,\Xi_n$ are possible forms of E.

If E be of the degree ν in ξ, and ν be an integer, the forms we have found for E are

$$\Xi_\nu, \quad \sqrt{\{(a^2+\xi)(b^2+\xi)\}}\,\Xi_{\nu-1}, \quad \sqrt{\{(b^2+\xi)(c^2+\xi)\}}\,\Xi_{\nu-1},$$
$$\sqrt{\{(c^2+\xi)(a^2+\xi)\}}\,\Xi_{\nu-1}.$$

We have found also that there are $\nu+1$ different functions of the first type, and ν of each of the others; so that there are $4\nu+1$ in all.

If $\nu = n + \frac{1}{2}$, where n is an integer, the forms found for E are

$$\sqrt{(a^2 + \xi)}\,\Xi_n, \quad \sqrt{(b^2 + \xi)}\,\Xi_n, \quad \sqrt{(c^2 + \xi)}\,\Xi_n,$$
$$\sqrt{\{(a^2 + \xi)(b^2 + \xi)(c^2 + \xi)\}}\,\Xi_{n-1}.$$

Also, there are $n + 1$ functions of each of the first three types, and n of the fourth, so that there are $4n + 3$; that is, $4\nu + 1$, in all.

Hence, in any case, there are $4\nu + 1$ determinable functions of ξ of the degree ν, any one of which may be taken for E in order to satisfy (5).

It is plain that if H be the same function of η, and H' of ζ, as E is of ξ, the product $CEHH'$, where C is an arbitrary constant, will then satisfy Laplace's equation.

169. **Ellipsoidal Harmonics which vanish at Infinity.** — The functions considered in the preceding Article do not vanish at infinity, and are therefore unfit to represent the potential of a finite mass in space outside itself. The form of the differential equation for E enables us, however, to obtain another function which will fulfil this condition.

In fact, if

$$\left(\mathfrak{X}\frac{d}{d\xi}\right)^2 y = (m\xi + j)\,y, \tag{10}$$

we have

$$\frac{d^2y}{d\xi^2} + \frac{\dfrac{d}{d\xi}\left(\mathfrak{X}^2\right)}{2\mathfrak{X}^2}\frac{dy}{d\xi} - \frac{m\xi + j}{\mathfrak{X}^2}\,y = 0\,;$$

but, as was shown in Art. 156, by assuming $y = y_1 u$, if y_1 be a solution of the equation

$$\frac{d^2y}{dx^2} + X_1\frac{dy}{dx} + X_2 y = 0,$$

then

$$Cy_1\int\frac{e^{-\int X_1 dx}}{y_1^{\,2}}\,dx$$

is also a solution.

In the present case, $\int X_1 dx = \log \mathfrak{X}$; and, therefore, if E be a solution of (10), so also is

$$CE \int \frac{d\xi}{\mathfrak{X} E^2}.$$

If now we take for E one of the forms found in the last Article, by writing E as the product of ξ^ν and a series of descending powers of ξ, it is plain that

$$E \int \frac{d\xi}{\mathfrak{X} E^2}$$

vanishes when ξ is infinite.

Hence we see that, if V denote the potential inside an ellipsoid, whose semi-axes are a, b, c, of a distribution of mass on its surface, and V' the potential of the same distribution in external space, and if $V = CEHH'$, then

$$V' = C'EHH' \int_\xi^\infty \frac{d\xi}{\mathfrak{X} E^2}, \tag{9}$$

where

$$C' \int_0 \frac{d\xi}{\mathfrak{X} E^2} = C.$$

170. **Ellipsoidal Harmonics expressed as functions of Cartesian Coordinates.**—If Ξ_n be a rational and integral function of ξ whose factors are $\xi - a_1$, $\xi - a_2$, ... $\xi - a_n$, and $E = \Xi_n$, then

$$CEHH' = \left.\begin{array}{l} (a^2 + a_1)(b^2 + a_1)(c^2 + a_1)\left(\dfrac{x^2}{a^2 + a_1} + \dfrac{y^2}{b^2 + a_1} + \dfrac{z^2}{c^2 + a_1} - 1\right) \\ (a^2 + a_2)(b^2 + a_2)(c^2 + a_2)\left(\dfrac{x^2}{a^2 + a_2} + \dfrac{y^2}{b^2 + a_2} + \dfrac{z^2}{c^2 + a_2} - 1\right) \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ (a^2 + a_n)(b^2 + a_n)(c^2 + a_n)\left(\dfrac{x^2}{a^2 + a_n} + \dfrac{y^2}{b^2 + a_n} + \dfrac{z^2}{c^2 + a_n} - 1\right) \end{array}\right\}. \tag{11}$$

For the expression

$$(a^2+u)(b^2+u)(c^2+u)\left(\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}-1\right)$$

vanishes when $u=\xi$, or $u=\eta$, or $u=\zeta$, where ξ, η, ζ are the elliptic coordinates of the point whose Cartesian coordinates are x, y, z.

Hence, whatever be u, we have

$$(a^2+u)(b^2+u)(c^2+u)\left(\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}-1\right)$$
$$= K_1(\xi-u)(\eta-u)(\zeta-u),$$

and therefore,

$$(a^2+a_1)(b^2+a_1)(c^2+a_1)\left(\frac{x^2}{a^2+a_1}+\frac{y^2}{b^2+a_1}+\frac{z^2}{c^2+a_1}-1\right)$$
$$= K_2(\xi-a_1)(\eta-a_1)(\zeta-a_1),$$

where K_1 and K_2 denote quantities independent of the coordinates.

Hence, if we denote the right-hand member of the equation (Ω, we have

$$\Omega = C(\xi \quad . (\xi-a_n)(\eta-a_1)(\eta-a_2)\dots(\eta-a_n)$$
$$(\zeta-a_1)(\zeta-a_2)\dots(\zeta-a_n) = CEHH'.$$

ssions for x, y, and z given by (6) we
ıat when

$$\left.\begin{aligned} E &= \surd \quad \xi\,\Xi_n \\ CEHH' &= \surd\{(a^2-b^2)(a^2-c^2)\}\,x\Omega \end{aligned}\right\}, \quad (12)$$

$$\left.\begin{aligned} E &= \surd\{(a^2+\xi)(b^2+\xi)\}\,\Xi_n \\ CEHH' &= (a^2-b^2)\surd\{(b^2-c^2)(c^2-a^2)\}\,xy\Omega \end{aligned}\right\}, \quad (13)$$

$$\left.\begin{aligned} E &= \surd\{(a^2+\xi)(b^2+\xi)(c^2+\xi)\}\,\Xi_n \\ CEHH' &= (a^2-b^2)(a^2-c^2)(b^2-c^2)\sqrt{-1}\,xyz\Omega \end{aligned}\right\}. \quad (14)$$

We have seen that there are $4\nu+1$ ellipsoidal harmonics of degree ν in ξ, that is, of degree 2ν in Cartesian coordinates.

A rational and integral function of x, y, z, of the degree i, contains

$$\frac{(i+1)(i+2)(i+3)}{6} \text{ constants;}$$

but if it satisfies Laplace's equation, these constants must satisfy

$$\frac{(i-1)\,i\,(i+1)}{6} \text{ equations,}$$

and therefore such a function contains only $(i+1)^2$ independent constants. Now, if we take all the different ellipsoidal harmonics from the degree 0 up to the degree i in x, y, z, or $\frac{1}{2}i$ in ξ, we have $1+3+5 \ldots . + 2i+1$ in all; the sum of this series is $(i+1)^2$.

Hence, as each harmonic may be multiplied by an arbitrary constant, we can express any rational and integral function of x, y, z of the degree i, which satisfies Laplace's equation, by a series of ellipsoidal harmonics, whose degrees in x, y, z range from i to 0.

At the surface of an ellipsoid of the confocal system *any* rational and integral function of x, y, z can be expressed as a series of ellipsoidal harmonics.

For, if a, b, c be the semi-axes of the ellipsoid, at its surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$$

and therefore, by multiplication, the degree of any function of x, y, z can be increased by 2 without altering its value. Hence a rational and integral function of the degree i can be reduced to two homogeneous functions of the degrees i and $i-1$. Of these, the first contains $\frac{(i+1)(i+2)}{2}$ independent constants, and the second $\frac{i(i+1)}{2}$. Hence the two together contain $(i+1)^2$ independent constants, and can therefore be expressed as a series of ellipsoidal harmonics.

171. **Surface Integral of Product of Harmonics.**—If $V_1 = E_1 H_1 H'_1$, and $V_2 = E_2 H_2 H'_2$, and S and S' denote two confocal ellipsoids of the system, whose normals drawn into the space between them are ν and ν', by Green's theorem, we have

$$\int V_1 \frac{dV_2}{d\nu} dS + \int V_1 \frac{dV_2}{d\nu'} dS' = \int V_2 \frac{dV_1}{d\nu} dS + \int V_2 \frac{dV_1}{d\nu'} dS',$$

also $d\nu = \dfrac{d\xi}{2p}$, where p is the central perpendicular on the tangent plane to S, and $d\nu' = -\dfrac{d\xi}{2p'}$, and therefore

$$2\int\left(V_1 \frac{dV_2}{d\xi} - V_2 \frac{dV_1}{d\xi}\right) p dS = 2\int\left(V_1 \frac{dV_2}{d\xi} - V_2 \frac{dV_1}{d\xi}\right) p' dS',$$

that is,

$$\int\left(E_1 \frac{dE_2}{d\xi} - E_2 \frac{dE_1}{d\xi}\right) H_1 H'_1 H_2 H'_2 \, p dS$$
$$= \int\left(E_1 \frac{dE_2}{d\xi} - E_2 \frac{dE_1}{d\xi}\right) H_1 H'_1 H_2 H'_2 \, p' dS'.$$

At corresponding points on the surfaces S and S' the coordinates η and ζ are the same, and therefore, so also are the values of H_1, H'_1, H_2, H'_2. Also, by Ex. 7, Art. 90, the volume elements pdS and $p'dS'$ are proportional to the products of the semi-axes of S and S', that is, to $\mathfrak{X}$ and $\mathfrak{X}'$. Hence, as E and $\dfrac{dE}{d\xi}$ are constant over the surface S, we have

$$\left\{\left(E_1 \frac{dE_2}{d\xi} - E_2 \frac{dE_1}{d\xi}\right)_{\xi}\right.$$
$$\left. - \frac{\mathfrak{X}'}{\mathfrak{X}}\left(E_1 \frac{dE_2}{d\xi} - E_2 \frac{dE_1}{d\xi}\right)_{\xi'}\right\} \int H_1 H_2 H'_1 H'_2 \, p dS = 0.$$

If we equate to zero the first factor of the left-hand side of this equation, we get

$$\mathfrak{X} E_1^2 d\left(\frac{E_2}{E_1}\right) = \mathfrak{X}' E'^2_1 d\left(\frac{E'_2}{E'_1}\right).$$

Since one surface may be taken as fixed and the other as variable, this equation is equivalent to

$$\mathfrak{X} E_1^2 \frac{d}{d\xi}\left(\frac{E_2}{E_1}\right) = \text{constant} = C,$$

whence
$$E_2 = CE_1 \int \frac{d\xi}{\mathfrak{X} E_1^2} + C'E_1.$$

Accordingly, either E_2 and E_1 differ only by a constant factor, or E_2 is the external harmonic corresponding to the internal E_1. In either case H_1 is the same as H_2, and H'_1 as H'_2. If we reject the alternatives considered above, we must have

$$\int H_1 H_2 H'_1 H'_2 \, p dS = 0,$$

and therefore,

$$\int V_1 V_2 \, p dS = 0.$$

Hence we conclude that *the surface integral of the product of two ellipsoidal harmonics and the central perpendicular on the tangent plane, taken over an ellipsoid of the confocal system, is zero, unless the two harmonics have a constant ratio to each other, or be a corresponding pair of harmonics, one internal and the other external.*

172. **Identity of Terms in equal Series.**—*If two series of internal or of external harmonics be equal to each other, each harmonic of one series must be identical with a corresponding harmonic of the other.*

To prove this, let the series

$$V_0 + V_1 + V_2 \ldots V_n = U_0 + U_1 \ldots + U_n;$$

multiply each side of this equation by $V_m p dS$, and integrate over the surface S, then all the integrals on the left-hand side vanish except $\int V_m^2 p dS$, and on the right-hand side they all vanish, unless $U_m = CV_m$, in which case we have

$$\int U_m V_m p dS = C \int V_m^2 p dS.$$

Hence $C = 1$, and a harmonic of the right-hand series is identical with V_m.

If two series of harmonics be equal throughout the whole of the space inside or outside an ellipsoid, both series must be composed of harmonics of the same kind, either internal or external, since an internal harmonic becomes infinite in external space at an infinite distance from the centre, and the differential coefficient of an external harmonic becomes infinite at the focal ellipse in the plane of xy.

It is easy to show by multiplication and integration over the surface S that, if two series of harmonics be equal at the surface of the ellipsoid S whose semi-axes are a, b, c, and one series be composed of internal harmonics, the other of external; then, if a term V_m of the first series be given by the equation

$$V_m = E_m H_m H'_m,$$

there must be a term U_m in the second such that

$$U_m = CE_m H_m H'_m \int_\xi^\infty \frac{d\xi}{\mathfrak{X} E_m^2},$$

where $$C \int_0^\infty \frac{d\xi}{\mathfrak{X} E_m^2} = 1.$$

173. **Density of Surface Distribution producing given Potential.**—If V denote the potential inside the ellipsoid a, b, c of a distribution of mass on its surface, and V' the potential in external space of the same distribution, and if

$$V = CEHH' \int_0^\infty \frac{d\xi}{\mathfrak{X} E^2},$$

we have seen, Art. 169, that

$$V' = CEHH' \int_\xi^\infty \frac{d\xi}{\mathfrak{X} E^2}.$$

In this case, if σ denote the density of the distribution, we have

$$\frac{dV}{d\nu} + \frac{dV'}{d\nu'} + 4\pi\sigma = 0;$$

but $$d\nu' = \frac{d\xi}{2p}, \quad d\nu = -\frac{d\xi}{2p},$$

whence

$$\frac{2\pi\sigma}{p} = \frac{dV}{d\xi} - \frac{dV'}{d\xi} = CHH'\frac{dE}{d\xi}\int_0^\infty \frac{d\xi}{\mathfrak{X}E^2}$$

$$- CHH'\frac{dE}{d\xi}\int_0^\infty \frac{d\xi}{\mathfrak{X}E^2} + C\frac{HH'}{\mathfrak{X}_0(E)_0};$$

that is,

$$2\pi\sigma = \frac{C}{abc\,(E)_0}pHH'. \tag{15}$$

If the potential due to the surface distribution whose density is σ be the sum of a number of harmonics V_0, V_1, V_2, &c., it may be shown in a similar manner that

$$\frac{2\pi abc}{p}\sigma = C_0 + \frac{C_1}{(E_1)_0}H_1H'_1 + \frac{C_2}{(E_2)_0}H_2H'_2 + \text{\&c.}, \tag{16}$$

where

$$V_0 = C_0\int_0^\infty \frac{d\xi}{\mathfrak{X}}, \quad V_1 = C_1E_1H_1H'_1\int_0^\infty \frac{d\xi}{\mathfrak{X}E_1^2}, \ \text{\&c.}$$

When σ is assigned, (16) enables us to determine the functions H_1, H'_1, &c., and from thence V_1, &c.

174. **Potential of Homœoid.**—As an example of the mode of procedure described in the preceding article, we may find the potential of a homœoid. Here σ varies as p, and $\frac{\sigma}{p}$ is constant, whence

$$V = C_0\int_0^\infty \frac{d\xi}{\mathfrak{X}}, \quad V' = C_0\int_\xi^\infty \frac{d\xi}{\mathfrak{X}}.$$

To determine C_0 we have

$$\frac{dV'}{d\nu'} = 2p\frac{dV'}{d\xi} = -\frac{2pC_0}{\mathfrak{X}_0}.$$

Hence, $$\frac{2C_0}{abc}\int pdS = \int NdS = 4\pi M,$$

where M is the total mass of the homœoid, but $\int pdS = 4\pi abc$,

and therefore $2C_0 = M$; accordingly, V', the potential of the homœoid in external space, is given by the equation

$$V' = \frac{M}{2}\int_\xi^\infty \frac{d\xi}{\mathfrak{X}}.$$

This agrees with the result found in Ex. 3, Art. 75.

175. **Ellipsoidal Harmonics of the Second Degree in the Coordinates.**—The forms of E which correspond to functions of the second degree in the coordinates are

$$\sqrt{\{(a^2+\xi)(b^2+\xi)\}}, \quad \sqrt{\{(b^2+\xi)(c^2+\xi)\}}, \quad \sqrt{\{(c^2+\xi)(a^2+\xi)\}},$$

and $$(\xi - \alpha).$$

There are two functions of the last form, $\xi - \alpha_1$, and $\xi - \alpha_2$. We proceed to determine the values of α_1 and α_2.

By (5), Art. 168, we have

$$(m\xi + j)(\xi - \alpha) = \left(\mathfrak{X}\frac{d}{d\xi}\right)^2(\xi - \alpha)$$

$$= \tfrac{3}{2}\xi^2 + (a^2+b^2+c^2)\xi + \tfrac{1}{2}(a^2b^2 + b^2c^2 + c^2a^2);$$

whence

$$m = \tfrac{3}{2}, \quad j - m\alpha = a^2+b^2+c^2, \quad -j\alpha = \tfrac{1}{2}(a^2b^2+b^2c^2+c^2a^2).$$

Eliminating m and j, we obtain

$$3\alpha^2 + 2(a^2+b^2+c^2)\alpha + a^2b^2 + b^2c^2 + c^2a^2 = 0. \qquad (17)$$

Hence,

$$\left.\begin{aligned}\alpha_1 &= \frac{-(a^2+b^2+c^2) + \sqrt{(a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2)}}{3}\\ \alpha_2 &= \frac{-(a^2+b^2+c^2) - \sqrt{(a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2)}}{3}\end{aligned}\right\}. \qquad (18)$$

177. **Potential of a Focaloid.**—We have seen, Art. 83, that, for a focaloid distribution of mass, the surface density σ is given by the equation

$$4\pi\sigma = \frac{2K}{p},$$

where

$$K = \frac{2\pi\rho a^2 b^2 c^2}{a^2 b^2 + b^2 c^2 + c^2 a^2},$$

... each product its value the density of the ... thus, $2f_1 yz$ becomes by ρ. By Ar... $2f_1$ space is $\rho)(b^2 - c^2)(c^2 - a^2)(c^2 - b^2)\}$

$$\times \sqrt{\{(b^2+\xi)(b^2+\eta)(b^2+\zeta)(c^2+\xi)(c^2+\eta)(c^2+\zeta)\}};$$

and the remaining two products are reduced to expressions of a similar kind.

We may next assume

$$a_1 x^2 + a_2 y^2 + a_3 z^2$$
$$= C_0 + K_1(\xi - a_1)(\eta - a_1)(\zeta - a_1) + K_2(\xi - a_2)(\eta - a_2)(\zeta - a_2),$$

where a_1 and a_2 have the values given by (18). In this manner we get

of proce... find the po... constant,

$$a_1 x^2 ... = C_0 \left.\begin{array}{l} + C_1(a^2+a_1)(b^2+a_1)(c^2+a_1) \\ \quad \times \left\{\dfrac{x^2}{a^2+a_1} + \dfrac{y^2}{b^2+a_1} + \dfrac{z^2}{c^2+a_1} - 1\right\} \\ + C_2(a^2+a_2)(b^2+a_2)(c^2+a_2) \\ \quad \times \left\{\dfrac{x^2}{a^2+a_2} + \dfrac{y^2}{b^2+a_2} + \dfrac{z^2}{c^2+a_2} - 1\right\} \end{array}\right\}.$$

Hence we have

$$\left.\begin{array}{l} C_0 - C_1(a^2+a_1)(b^2+a_1)(c^2+a_1) - C_2(a^2+a_2)(b^2+a_2)(c^2+a_2) = 0, \\ a_1 = C_1(b^2+a_1)(c^2+a_1) + C_2(b^2+a_2)(c^2+a_2), \\ a_2 = C_1(c^2+a_1)(a^2+a_1) + C_2(c^2+a_2)(a^2+a_2). \end{array}\right\} \quad (19)$$

By substituting the equivalent expressions in x, y, and z for the functions

$$(\xi - \alpha_1)(\eta - \alpha_1)(\zeta - \alpha_1) \quad \text{and} \quad (\xi - \alpha_2)(\eta - \alpha_2)(\zeta - \alpha_2),$$

the potential U becomes of the form

$$a_0 + a_1 x^2 + a_2 y^2 + a_3 z^2,$$

where a_0, &c., are constants.

$$C_1 = \frac{a_1 a^2 - a_2 b^2 + \ldots}{(a^2 - b^2)(c^2 + \alpha_1)(\alpha_1 - \alpha_2)},$$

of Attraction of Focaloid and

...om the form of U it follows ...l space due to a

At the surface of the ellipsoid a, b, c, the q... of force function $a_1 x^2 + a_2 y^2 + a_3 z^2$ can be reduced to ellips... harmonics, whatever be the values of a_1, a_2, and a_3. Fo..., in this case, we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0;$$

and therefore we may substitute for the given function the expression

$$\left(a_1 + \frac{\varepsilon}{a^2}\right)x^2 + \left(a_2 + \frac{\varepsilon}{b^2}\right)y^2 + \left(a_3 + \frac{\varepsilon}{c^2}\right)z^2 - \varepsilon.$$

We can then determine ϵ so as to satisfy the equation

$$a_1 + \frac{\varepsilon}{a^2} + a_2 + \frac{\varepsilon}{b^2} + a_3 + \frac{\varepsilon}{c^2} = 0,$$

and putting $a_1 + \dfrac{\varepsilon}{a^2}$ for a_1, and $a_2 + \dfrac{\varepsilon}{b^2}$ for a_2, proceed in the same manner as before. In this case, the right-hand side of the first of the equations (19) is not zero but $-\varepsilon$, so that

$$C_0 = C_1(a^2 + \alpha_1)(b^2 + \alpha_1)(c^2 + \alpha_1) + C_2(a^2 + \alpha_2)(b^2 + \alpha_2)(c^2 + \alpha_2) - \epsilon.$$

In like manner,

$$Y = \beta y \int_\xi^\infty \frac{d\xi}{\mathfrak{X}(b^2+\xi)}, \quad Z = \gamma z \int_\xi^\infty \frac{d\xi}{\mathfrak{X}(c^2+\xi)}.$$

At a great distance r from the centre, X, Y, and Z tend towards the values

$$\frac{M}{r^3}x, \quad \frac{M}{r^3}y, \quad \text{and} \quad \frac{M}{r^3}z,$$

where M denotes the mass of the ellipsoid.

If we expand in descending powers of ξ the functions under the integral sign in the expressions for X, Y, and Z, and integrate, we find that X tends towards $\frac{2}{3}\alpha x\xi^{-\frac{3}{2}}$, and that Y tends towards $\frac{2}{3}\beta y\xi^{-\frac{3}{2}}$, and Z towards $\frac{2}{3}\gamma z\xi^{-\frac{3}{2}}$.

Hence we have $\alpha = \beta = \gamma = \frac{3}{2}M$, and we get

$$\left.\begin{aligned} X &= \tfrac{3}{2}Mx \int_\xi^\infty \frac{d\xi}{(a^2+\xi)\mathfrak{X}}, \\ Y &= \tfrac{3}{2}My \int_\xi^\infty \frac{d\xi}{(b^2+\xi)\mathfrak{X}}, \\ Z &= \tfrac{3}{2}Mz \int_\xi^\infty \frac{d\xi}{(c^2+\xi)\mathfrak{X}}. \end{aligned}\right\} \tag{22}$$

179. **Potential of Ellipsoid in External Space.**— If V' denote the potential of the ellipsoid in external space,

$$V' = -\int (X dx + Y dy + Z dz) + \text{constant}.$$

Integrating by parts, we find

$$\int dx\, x \int_\xi^\infty \frac{d\xi}{(a^2+\xi)\mathfrak{X}} = \frac{x^2}{2}\int_\xi^\infty \frac{d\xi}{(a^2+\xi)\mathfrak{X}} + \int \frac{x^2}{2} \frac{\frac{d\xi}{dx}dx}{(a^2+\xi)\mathfrak{X}}$$

$$= \frac{x^2}{2}\int_\xi^\infty \frac{d\xi}{(a^2+\xi)\mathfrak{X}} + \int^\xi \frac{x^2}{2}\frac{d\xi}{(a^2+\xi)\mathfrak{X}},$$

where in the second integral x is to be regarded as a function of ξ, y, and z; and y and z are to be looked upon as constant in the integration.

Similar results hold good for the functions contained in Y and Z. Hence we have

$$V' = -\tfrac{3}{4}M\left\{x^2\int_\xi^\infty \frac{d\xi}{(a^2+\xi)\mathfrak{X}} + y^2\int_\xi^\infty \frac{d\xi}{(b^2+\xi)\mathfrak{X}}\right.$$
$$\left. + z^2\int_\xi^\infty \frac{d\xi}{(c^2+\xi)\mathfrak{X}} - \int_\xi^\infty\left(\frac{x^2}{a^2+\xi}+\frac{y^2}{b^2+\xi}+\frac{z^2}{c^2+\xi}\right)\frac{d\xi}{\mathfrak{X}}\right\}.$$

No constant is to be added, since the right-hand side of this equation vanishes at infinity. The three integrations involved in the last integral on the right-hand side are to be performed on three different hypotheses; but, as

$$\frac{x^2}{a^2+\xi}+\frac{y^2}{b^2+\xi}+\frac{z^2}{c^2+\xi} = 1,$$

we have, finally,

$$V' = \tfrac{3}{4}M\left\{\int_\xi^\infty \frac{d\xi}{\mathfrak{X}} - x^2\int_\xi^\infty \frac{d\xi}{(a^2+\xi)\mathfrak{X}} - y^2\int_\xi^\infty \frac{d\xi}{(b^2+\xi)\mathfrak{X}} - z^2\int_\xi^\infty \frac{d\xi}{(c^2+\xi)\mathfrak{X}}\right\}. \tag{23}$$

We see that this result agrees with (11), Art. 87, remembering that in that equation x, y, z are not regarded as functions of u.

180. **Potential of Ellipsoid in its Interior.**—The components of force due to a focaloid at the outside of its surface, by (22), are

$$\tfrac{3}{2}Mx\int_0^\infty \frac{d\xi}{(a^2+\xi)\mathfrak{X}},\ \ \&c.;$$

that is, if we put

$$A = \tfrac{3}{2}M\int_0^\infty \frac{d\xi}{(a^2+\xi)\mathfrak{X}},\quad B = \tfrac{3}{2}M\int_0^\infty \frac{d\xi}{(b^2+\xi)\mathfrak{X}},$$
$$C = \tfrac{3}{2}M\int_0^\infty \frac{d\xi}{(c^2+\xi)\mathfrak{X}},$$

they are Ax, By, and Cz.

Hence, at any point inside the focaloid, if these components be denoted by X, Y, and Z, we have

$$X = \left(A - \frac{2K}{a^2}\right)x, \quad Y = \left(B - \frac{2K}{b^2}\right)y, \quad Z = \left(C - \frac{2K}{c^2}\right)z.$$

For these expressions satisfy Laplace's equation throughout the interior of the focaloid, and take the proper values at its inner surface.

Hence the potential U of the focaloid is given by the equation

$$U = -\tfrac{1}{2}(Ax^2 + By^2 + Cz^2) + K\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) + A_0,$$

where A_0 denotes an undetermined constant.

By Art. 83, if V denote the potential of the ellipsoid inside itself,

$$V - U = K\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right).$$

Hence,

$$V = K + A_0 - \tfrac{1}{2}(Ax^2 + By^2 + Cz^2).$$

At the surface of the ellipsoid, $V' = V$, and therefore, by (23), we have

$$K + A_0 = \tfrac{3}{4}M\int_0^\infty \frac{d\xi}{\mathfrak{X}};$$

accordingly,

$$V = \tfrac{3}{4}M\left\{\int_0^\infty \frac{d\xi}{\mathfrak{X}} - x^2\int_0^\infty \frac{d\xi}{(a^2+\xi)\mathfrak{X}} - y^2\int_0^\infty \frac{d\xi}{(b^2+\xi)\mathfrak{X}} - z^2\int_0^\infty \frac{d\xi}{(c^2+\xi)\mathfrak{X}}\right\}. \tag{24}$$

CHAPTER IX.

MAGNETIZED BODIES.

SECTION I.—*Constitution and Action of Magnets.*

181. **Magnet of Finite Dimensions.**—A magnetized body is composed of elements each of which is a magnetic particle (Art. 17). When such a body is placed in a uniform field of magnetic force, each particle is acted on by a couple, and the resultant of all these couples tends to bring the body into a position in which a certain line in the body is in the direction of the uniform force. When the body is in the position in which this couple is the greatest possible, its ratio to the force is the *magnetic moment of the body.*

If a body be composed of a number of infinitely thin parallel bars, magnetized at their extremities so that the pole strength of each bar is proportional to its orthogonal section, it is plain that the magnetic moment of the body is proportional to the sum of the products obtained by multiplying the length of each bar by the area of its orthogonal section—in other words, to the volume of the body. Hence we may assume that the magnetic moment of an element of a magnetized body is proportional to the volume of the element, and we may denote this magnetic moment by the expression $I d\mathfrak{S}$, where $d\mathfrak{S}$ denotes the volume of the element, and I the *intensity of magnetization.* This latter is defined as *the ratio of the magnetic moment of the element to its volume.* Magnetization is a directed quantity, and its direction is that of the parallel bar magnets which are regarded as composing the element whose magnetic axis is a line in this direction.

182. **Potential of Magnetized Body.**—By (28), Art. 54, the potential of one element of the body is

$$\frac{I \cos \varepsilon \, d\mathfrak{S}}{r^2}.$$

If x, y, z denote the coordinates of the element; ξ, η, ζ those of the point at which the potential is required; λ, μ, ν the direction-cosines of the magnetic axis of $d\mathfrak{S}$, we have

$$\cos \varepsilon = -\left(\lambda \frac{dr}{dx} + \mu \frac{dr}{dy} + \nu \frac{dr}{dz}\right).$$

The quantities $I\lambda$, $I\mu$, and $I\nu$ are termed the components of magnetization, and may be denoted by A, B, and C. If V denote the potential of the magnetized body, we have, then,

$$V = \iiint \left\{A \frac{d}{dx}\left(\frac{1}{r}\right) + B \frac{d}{dy}\left(\frac{1}{r}\right) + C \frac{d}{dz}\left(\frac{1}{r}\right)\right\} dx\,dy\,dz. \quad (1)$$

If l, m, n denote the direction-cosines of the normal to the surface S, which is the boundary of the body, we get, by integration,

$$V = \iint (lA + mB + nC) \frac{dS}{r} - \iiint \left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz}\right) \frac{d\mathfrak{S}}{r}. \quad (2)$$

Hence the potential of a magnetized body is equivalent to that of a volume-distribution, throughout the space occupied by the body, of mass whose density is

$$-\left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz}\right),$$

together with a distribution on the surface bounding the body whose density is

$$lA + mB + nC.$$

183. **Poisson's Equation.**—From the expression for the potential given by (2) we have

$$\nabla^2 V - 4\pi\left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz}\right) = 0 \qquad (3)$$

at any point inside the body.

In space outside the body Laplace's equation obviously holds good.

184. **Examples of Magnetized Bodies.**—As an example of a magnetized body, we may take a sphere magnetized in a uniform direction so that the magnetization at any point is a function of its distance from the centre. Here, if r' denote the distance from the centre of any point of the sphere whose coordinates are x', y', z'; V the magnetic potential of the sphere at an external point whose coordinates are x, y, z; and r the distance between the points xyz and $x'y'z'$, we have, the direction of magnetization being parallel to x,

$$V = \int f(r')\frac{d}{dx'}\left(\frac{1}{\mathrm{r}}\right)d\mathfrak{S}' = -\frac{d}{dx}\int f(r')\frac{d\mathfrak{S}'}{\mathrm{r}} = X,$$

where X denotes the component of force due to a sphere whose density at any point is $f(r')$ Hence

$$V = \frac{4\pi x}{r^3}\int_0^a f(r')\,r'^2\,dr',$$

where a denotes the radius of the sphere, and therefore the magnetic action of the sphere at an external point is the same as that of a small magnet at the centre whose magnetic moment is expressed by

$$4\pi\int_0^a r'^2 f(r')\,dr'.$$

If the magnetization be of uniform intensity I, the magnetic moment becomes

$$\frac{4\pi I}{3}a^3.$$

—By (28), If an ellipsoid be uniformly magnetized in the dire.. of its longest axis, the potential V, at an external point xyz, is given by the equation

$$V = I\int \frac{d}{dx'}\left(\frac{1}{\mathrm{r}}\right) d\mathfrak{S} = -I\frac{d}{dx}\int\frac{d\mathfrak{S}}{\mathrm{r}} = X,$$

where X denotes the component of force of a solid homogeneous ellipsoid whose density is I. Hence by (12), Art. 87, we have

$$V = 2\pi Iabcx\int_q^\infty \frac{du}{(a^2+u)^{\frac{3}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{1}{2}}}, \tag{4}$$

where q is the greatest root of the equation

$$\frac{x^2}{a^2+q} + \frac{y^2}{b^2+q} + \frac{z^2}{c^2+q} = 1.$$

(See Ex. 1, Art. 52.)

If the integral in (4) be denoted by $\mathfrak{X}$, and the corresponding integrals for the other two axes by $\mathfrak{Y}$ and $\mathfrak{Z}$, it is easy to see that the potential V of an ellipsoid, uniformly magnetized in a direction inclined to the axes at angles whose cosines are λ, μ, ν, is given by the equation

$$V = 2\pi Iabc(\lambda x\mathfrak{X} + \mu y\mathfrak{Y} + \nu z\mathfrak{Z}). \tag{5}$$

From (5) we can obtain the components of the magnetic force exerted by the ellipsoid at an external point. By differentiation we have

$$\frac{dV}{dx} = 2\pi Iabc\left(\lambda\mathfrak{X} + \lambda x\frac{d\mathfrak{X}}{dx} + \mu y\frac{d\mathfrak{Y}}{dx} + \nu z\frac{d\mathfrak{Z}}{dx}\right);$$

but

$$\frac{d\mathfrak{X}}{dx} = -\frac{1}{(a^2+q)^{\frac{3}{2}}(b^2+q)^{\frac{1}{2}}(c^2+q)^{\frac{1}{2}}}\frac{dq}{dx}.$$

If we denote the semi-axes of the ellipsoid passing through the point xyz and confocal with the given ellipsoid by a', b', c', we have

$$\frac{2x}{a'^2} - \left(\frac{x^2}{a'^4} + \frac{y^2}{b'^4} + \frac{z^2}{c'^4}\right)\frac{dq}{dx} = 0.$$

Hence, if p' denote the central perpendicular on the tangent plane to the ellipsoid $a'b'c'$ at the point xyz, we have

$$\frac{dq}{dx} = 2x\frac{p'^2}{a'^2}.$$

Hence
$$\frac{d\mathfrak{X}}{dx} = -\frac{2xp'^2}{a'^5 b' c'}.$$

In like manner,

$$\frac{d\mathfrak{Y}}{dx} = -\frac{2xp'^2}{a'^3 b'^3 c'}, \quad \frac{d\mathfrak{Z}}{dx} = -\frac{2xp'^2}{a'^3 b' c'^3};$$

accordingly, by substitution, we obtain

$$\frac{dV}{dx} = 2\pi Iabc\lambda\mathfrak{X} - \frac{4\pi Iabc}{a'b'c'}\frac{p'x}{a'^2}\left(\frac{\lambda p'x}{a'^2} + \frac{\mu p'y}{b'^2} + \frac{\nu p'z}{c'^2}\right).$$

If ϖ_1, ϖ_2, ϖ_3, and ϑ denote the angles which the normal to the ellipsoid $a'b'c'$ at the point xyz makes with the axes, and with the direction of magnetization, we have

$$\frac{p'x}{a'^2} = \cos\varpi_1, \text{ \&c.};$$

and
$$\cos\vartheta = \lambda\cos\varpi_1 + \mu\cos\varpi_2 + \nu\cos\varpi_3.$$

Hence we obtain

$$\frac{dV}{dx} = 2\pi Iabc\lambda\mathfrak{X} - \frac{4\pi Iabc}{a'b'c'}\cos\vartheta\cos\varpi_1;$$

and, since the components of magnetic force α, β, γ are expressed by $-\frac{dV}{dx}$, &c., we have

$$\left.\begin{aligned} \alpha &= \frac{4\pi Iabc}{a'b'c'}\cos\vartheta\cos\varpi_1 - 2\pi Iabc\lambda\mathfrak{X}', \\ \beta &= \frac{4\pi Iabc}{a'b'c'}\cos\vartheta\cos\varpi_2 - 2\pi Iabc\mu\mathfrak{Y}', \\ \gamma &= \frac{4\pi Iabc}{a'b'c'}\cos\vartheta\cos\varpi_3 - 2\pi Iabc\nu\mathfrak{Z}'. \end{aligned}\right\} \qquad (6)$$

From equations (6) it appears that the force exerted by the magnetized ellipsoid abc at an external point P is the resultant of two forces of which one is in the direction of the normal at P to the ellipsoid $a'b'c'$, and is expressed by

$$\frac{4\pi Iabc}{a'b'c'}\cos\vartheta;$$

the other is the force due to a homogeneous solid ellipsoid, coinciding with abc, at the point Q in which a line drawn from the centre in the direction of the magnetization meets the surface of the ellipsoid $a'b'c'$, the density of the solid ellipsoid, supposed attractive, being $\frac{I}{R}$, where R denotes the distance of Q from the centre.

185. **Potential of Magnetized Body expressed as Sum of Force Components.**—Adopting the notation of Art. 182, we have, by (1),

$$V = -\frac{d}{d\xi}\int\frac{Ad\mathfrak{S}}{\mathrm{r}} - \frac{d}{d\eta}\int\frac{Bd\mathfrak{S}}{\mathrm{r}} - \frac{d}{d\zeta}\int\frac{Cd\mathfrak{S}}{\mathrm{r}}. \qquad (7)$$

Hence, if we suppose three bodies geometrically identical with the magnetized body, and having for densities A, B, and C, the magnetic potential is equal to the sum of the force components exercised by the first body parallel to the axis of x, by the second parallel to the axis of y, and by the third parallel to the axis of z.

186. **Magnetic Force.**—The differential coefficients of the potential with their signs changed are termed the components of magnetic force. Outside the magnetized body these are the actual components of the force which the body would exert on a north magnetic pole of unit intensity.

Inside the body the actual force due to the body is indeterminate. In order to imagine that such a force should act, we must suppose a small cavity inside the body, and, in the case of a magnetized body, the force depends on the shape of this cavity.

The components of the magnetic force are usually denoted by the letters α, β, γ.

It is easy to see that the normal component of the magnetic force as defined above is not continuous when we pass from the outside to the inside of the magnetized body. This follows from the consideration that the normal component of that part of the force due to the surface distribution $lA + mB + nC$ is diminished by $4\pi(lA + mB + nC)$.

187. **Magnetic Induction.**—We can obtain a vector quantity whose components satisfy the solenoidal condition, and whose normal component at the boundary of the magnet is continuous, by adding to each component of magnetic force the corresponding component of magnetization multiplied by 4π. This vector quantity is termed the *magnetic induction*, and its components are usually denoted by the letters a, b, c. We have, then,

$$a = \alpha + 4\pi A, \quad b = \beta + 4\pi B, \quad c = \gamma + 4\pi C. \qquad (8)$$

Outside the magnet $a = \alpha$, $b = \beta$, $c = \gamma$; and

$$\frac{da}{d\xi} + \frac{db}{d\eta} + \frac{dc}{d\zeta} = -\nabla^2 V = 0.$$

Inside the magnet

$$\frac{da}{d\xi} + \frac{db}{d\eta} + \frac{dc}{d\zeta} = -\nabla^2 V + 4\pi\left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz}\right),$$

and, by (3), the right-hand member of this equation is zero; accordingly, a, b, and c always fulfil the solenoidal condition.

At the surface bounding the magnet, in passing from a point outside to a point inside, $l\alpha + m\beta + n\gamma$ is diminished by $4\pi(lA + mB + nC)$; but outside the surface $la + mb + nc$ is the same as $l\alpha + m\beta + n\gamma$, and inside the former exceeds the latter by $4\pi(lA + mB + nC)$.

Hence, in passing through the surface, the value of $la + mb + nc$ is unaltered.

It is now easy to see that the surface integral

$$\int (la + mb + nc)\, dS$$

taken over any closed surface is zero.

If the surface be altogether outside or altogether inside the magnetic body, this follows from taking the volume integral of

$$\frac{da}{d\xi} + \frac{db}{d\eta} + \frac{dc}{d\zeta}.$$

If the surface S be partly outside and partly inside the magnet, the enclosed volume is divided into two parts by the intercepted portion of the surface of the magnet. Through each of these parts the integration may be effected, and in consequence of the continuity of the normal component of magnetic induction, the two surface integrals which are taken over the portion of the magnet surface are equal in magnitude and opposite in algebraical sign, and therefore the surface integral of induction over the closed surface S is zero.

188. **Magnetic Force and Magnetic Induction regarded as Forces.**—If we imagine a small cylindrical cavity whose axis is in the direction of magnetization, and a north magnetic pole of unit intensity placed at the middle point of this axis, the actual force acting on this pole is the magnetic force when the cylinder is long and narrow, and the magnetic induction when the cylinder is short and broad.

As the cavity is supposed to be small, the removal of the volume distribution with which it was occupied produces no sensible change in the force acting on the magnet-pole, and this force is therefore due to the volume distribution throughout the magnet, the surface distribution on its boundary, and to the surface distribution on the surface bounding the cavity.

In the case of a cylinder parallel to the magnetization axis, $lA + mB + nC$ is zero except at the plane ends, where it is $-I$ at the positive end, and $+I$ at the negative. By (3), Art. (14), the force due to the surface-distribution is, therefore,

$$4\pi I \left\{1 - \frac{c}{\sqrt{(a^2 + c^2)}}\right\}$$

in the direction of magnetization, where c denotes the semi-axis of the cylinder, and a its radius.

When c is large compared with a, this expression becomes zero; and when a is large compared with c it becomes $4\pi I$. Hence, in the first case, the components of the total force acting on the magnet-pole are α, β, γ; and in the second $\alpha + 4\pi A, \quad \beta + 4\pi B, \quad \gamma + 4\pi C$.

189. **Energy due to Magnet.**—When a magnet is placed in an independent field of force, if V denote the potential of the field at any point where there is a south pole of strength $\mathfrak{M}$, the energy due to the presence of this pole is $-\mathfrak{M}V$, and that due to the corresponding north pole is

$$\mathfrak{M}\left(V + \frac{dV}{dh}dh\right),$$

where dh is the axis of the particle whose poles are $\mathfrak{M}$ and $-\mathfrak{M}$. Hence the energy due to the particle is

$$\mathfrak{M}\frac{dV}{dh}dh.$$

If λ, μ, ν be the direction-cosines of dh, we have

$$\frac{dV}{dh} = \lambda\frac{dV}{dx} + \mu\frac{dV}{dy} + \nu\frac{dV}{dz};$$

also, $\mathfrak{M}dh = Id\mathfrak{S}$, and therefore

$$\mathfrak{M}\frac{dV}{dh}dh = \left(A\frac{dV}{dx} + B\frac{dV}{d\cdot} + C\frac{dV}{}\right.$$

Consequently, if W denote the energy due to the presence of a magnet in an independent field of force,

$$W = \int \left(A \frac{dV}{dx} + B \frac{dV}{dy} + C \frac{dV}{dz} \right) d\mathfrak{S}, \tag{9}$$

where A, B, C denote the components of magnetization of the magnet at any point where the potential of the field is V.

190. **Energy of Magnetic System.**—When the field of force is due to the magnets which are present, it is plain that if the magnetization be everywhere increased in the same ratio, the potential is likewise increased in this ratio. Hence, by reasoning similar to that employed in Art. 50, we see that, if W denote the energy of a magnetic system, and V its potential at any point, we have

$$W = \tfrac{1}{2} \iiint \left(A \frac{dV}{dx} + B \frac{dV}{dy} + C \frac{dV}{dz} \right) dx\, dy\, dz. \tag{10}$$

If we integrate by parts the expression for W given by (10), we get

$$\begin{aligned} W &= \tfrac{1}{2} \Sigma \left\{ \int V(lA + mB + nC)\, dS - \int V \left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz} \right) d\mathfrak{S} \right\} \\ &= -\frac{1}{8\pi} \left\{ \Sigma \int V \left(\frac{dV}{d\nu} + \frac{dV}{d\nu'} \right) dS + \int V \nabla^2 V\, d\mathfrak{S} \right\} \\ &= \frac{1}{8\pi} \int (\alpha^2 + \beta^2 + \gamma^2)\, d\mathfrak{S}, \end{aligned} \tag{11}$$

where the last two integrals are taken throughout the whole of space.

191. **Vector Potential of Magnetic Induction.**—We have seen, Art. 187, that a, b, c, the components of mag- voluin induction, fulfil the solenoidal condition throughout the sensible ok space, and that the surface integral of induction this force is losed surface is zero. From hence it follows that out the magnet, tegral has the same value for any two surface to the surface distribution boundary.

Hence the integral of induction taken over a surface-sheet S must be expressible as a line integral taken round the curve s which is the boundary of S. We have, therefore, an equation of the form

$$\iint (la + mb + nc)\, dS = \int \left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right) ds.$$

The directed magnitude of which F, G, H are the components is called the *vector potential of magnetic induction.*

192. **Stokes's Theorem.**—If u, v, w denote three functions of the coordinates, Stokes's theorem is expressed by the equation

$$\iint \left\{l\left(\frac{dw}{dy} - \frac{dv}{dz}\right) + m\left(\frac{du}{dz} - \frac{dw}{dx}\right) + n\left(\frac{dv}{dx} - \frac{du}{dy}\right)\right\} dS$$
$$= \int \left(u\frac{dx}{ds} + v\frac{dy}{ds} + w\frac{dz}{ds}\right) ds, \qquad (12)$$

where S is a surface-sheet, and s the curve which forms its boundary.

To prove this, we observe that the terms in the surface integral which contain w may be written

$$\iint \left(\frac{dw}{dy} - \frac{m}{l}\frac{dw}{dx}\right) l dS, \quad \text{or} \quad \iint \left(\frac{dw}{dy} - \frac{m}{l}\frac{dw}{dx}\right) dy\, dz.$$

In this double integral x is regarded as a function of y and z given by the equation of the surface S.

If $\dfrac{d}{dy} w$ denote the differential coefficient of w taken on this hypothesis, we have

$$\frac{d}{dy} w = \frac{dw}{dy} + \frac{dw}{dx}\frac{dx}{dy};$$

but the differential equation of the surface may be written

$$l\, dx + m\, dy + n\, dz = 0,$$

and therefore $$\frac{dx}{dy} = -\frac{m}{l}.$$

Hence
$$\frac{d}{dy}w = \frac{dw}{dy} - \frac{m}{l}\frac{dw}{dx},$$

and
$$\iint\left(l\frac{dw}{dy} - m\frac{dw}{dx}\right)dS = \iint\frac{d}{dy}w\,dy\,dz = \int w\frac{dz}{ds}ds.$$

If the terms containing u and v in the double integral be treated in a similar manner, we obtain the right-hand member of (12). If the axes be drawn in the usual manner so that counter-clockwise rotations round x, y, and z bring y to z, z to x, and x to y, equation (12) shows that the direction of integration round s is counter-clockwise as viewed from the positive end of the normal to S.

If the surface-sheet S be contained between two curves, the surface integral is equal to the difference between two line integrals.

193. **Determination of Vector Potential.**—It follows from Stokes's theorem that as a consequence of (12) we may assume

$$\left.\begin{aligned} a &= \frac{dH}{d\eta} - \frac{dG}{d\zeta}, \\ b &= \frac{dF}{d\zeta} - \frac{dH}{d\xi}, \\ c &= \frac{dG}{d\xi} - \frac{dF}{d\eta}, \end{aligned}\right\} \tag{13}$$

where ξ, η, ζ denote the current coordinates. If F_1, G_1, H_1 be three functions of ξ, η, ζ satisfying these equations, it is plain that they will be satisfied also by

$$F_1 + \frac{d\phi}{d\xi},\quad G_1 + \frac{d\phi}{d\eta},\quad H_1 + \frac{d\phi}{d\zeta},$$

where ϕ is any function of the coordinates.

We see, then, that equations (13) are not sufficient to determine F, G, and H, and we may assume

$$\frac{dF}{d\xi} + \frac{dG}{d\eta} + \frac{dH}{d\zeta} = 0. \tag{14}$$

From (13) we have

$$\frac{db}{d\xi} - \frac{da}{d\eta} = \frac{d^2F}{d\xi d\zeta} + \frac{d^2G}{d\eta d\zeta} - \left(\frac{d^2H}{d\xi^2} + \frac{d^2H}{d\eta^2}\right)$$
$$= \frac{d}{d\zeta}\left(\frac{dF}{d\xi} + \frac{dG}{d\eta} + \frac{dH}{d\zeta}\right) - \nabla^2 H;$$

whence

$$\nabla^2 H + 4\pi\left(\frac{dB}{d\xi} - \frac{dA}{d\eta}\right) = 0, \tag{15}$$

since

$$\frac{d\beta}{d\xi} - \frac{da}{d\eta} = 0.$$

Equation (15) is similar in form to that for determining the potential of an attracting mass. Hence apparently we have

$$H = \iiint\left(\frac{dB}{dx} - \frac{dA}{dy}\right)\frac{d\mathfrak{S}}{\mathrm{r}},$$

the integral being taken through the whole of the magnetized body. This integral is, however, indeterminate, as at the surface A, B, and C are discontinuous, and their differential coefficients in the direction of the normal infinite.

If we integrate by parts inside the boundary of the magnet, we get for H the expression

$$\int\left(lB - mA\right)dS + \int\left(A\frac{d}{dy} - B\frac{d}{dx}\right)\frac{1}{\mathrm{r}}\,d\mathfrak{S}.$$

We may therefore assume

$$\left.\begin{aligned} F &= \int\left(B\frac{d}{dz} - C\frac{d}{dy}\right)\frac{1}{\mathrm{r}}\,d\mathfrak{S} \\ G &= \int\left(C\frac{d}{dx} - A\frac{d}{dz}\right)\frac{1}{\mathrm{r}}\,d\mathfrak{S} \\ H &= \int\left(A\frac{d}{dy} - B\frac{d}{dx}\right)\frac{1}{\mathrm{r}}\,d\mathfrak{S} \end{aligned}\right\}, \tag{16}$$

provided these forms satisfy the differential equations (13) and (14).

It is easy to see that this is the case, for since

$$\frac{d\mathrm{r}}{dx} = -\frac{d\mathrm{r}}{d\xi},$$

and A, B, C are not functions of ξ, η, ζ, we have

$$\frac{dH}{d\eta} - \frac{dG}{d\zeta}$$

$$= \frac{d}{d\eta}\int\left(B\frac{d}{d\xi} - A\frac{d}{d\eta}\right)\frac{1}{\mathrm{r}}\,d\mathfrak{S} - \frac{d}{d\zeta}\int\left(A\frac{d}{d\zeta} - C\frac{d}{d\xi}\right)\frac{1}{\mathrm{r}}\,d\mathfrak{S}$$

$$= \frac{d}{d\xi}\left\{\frac{d}{d\eta}\int\frac{B}{\mathrm{r}}\,d\mathfrak{S} + \frac{d}{d\zeta}\int\frac{C}{\mathrm{r}}\,d\mathfrak{S}\right\} - \left(\frac{d^2}{d\eta^2} + \frac{d^2}{d\zeta^2}\right)\int\frac{A}{\mathrm{r}}\,d\mathfrak{S}$$

$$= -\frac{d}{d\xi}\int\left(A\frac{d}{dx} + B\frac{d}{dy} + C\frac{d}{dz}\right)\frac{1}{\mathrm{r}}\,d\mathfrak{S} - \nabla^2\int\frac{A}{\mathrm{r}}\,d\mathfrak{S}; \tag{17}$$

but, by (1), the first term in the right-hand member of (17) is a, and in space outside the magnet the remaining term is zero, and inside the magnet, when $\xi\eta\zeta$ coincides with xyz, its value is $4\pi A$. Hence we obtain

$$\frac{dH}{d\eta} - \frac{dG}{d\zeta} = a + 4\pi A = a. \tag{18}$$

We may write $\dfrac{dF}{d\xi}$ in the form

$$\frac{d^2}{d\xi d\eta}\int\frac{C}{\mathrm{r}}\,d\mathfrak{S} - \frac{d^2}{d\zeta d\xi}\int\frac{B}{\mathrm{r}}\,d\mathfrak{S};$$

and expressing $\dfrac{dG}{d\eta}$ and $\dfrac{dH}{d\zeta}$ in a similar manner, we see that

$$\frac{dF}{d\xi} + \frac{dG}{d\eta} + \frac{dH}{d\zeta}$$

vanishes identically.

194. **Vector Potential of Magnetic Particle.**—In the case of a magnetic particle equations (16) become

$$F = \mathfrak{M}\left(\mu \frac{d}{dz} - \nu \frac{d}{dy}\right)\frac{1}{\mathrm{r}}, \text{ \&c.,}$$

where $\mathfrak{M}$ denotes the magnetic moment of the particle, and λ, μ, ν the direction-cosines of its axis.

If θ_1, θ_2, θ_3 denote the direction-cosines of r, we have

$$\frac{d}{dz}\frac{1}{\mathrm{r}} = \frac{\zeta - z}{\mathrm{r}^3} = \frac{\theta_3}{\mathrm{r}^2},$$

whence

$$F = \mathfrak{M}\frac{\mu\theta_3 - \nu\theta_2}{\mathrm{r}^2}, \text{ \&c.;}$$

but

$$\mu\theta_3 - \nu\theta_2 = \sin\epsilon\cos\vartheta_1,$$

where ϑ_1 denotes the angle which a perpendicular to r and the magnetic axis makes with the axis of x, and ϵ the angle between r and the magnetic axis. Hence

$$F = \frac{\mathfrak{M}\sin\epsilon}{\mathrm{r}^2}\cos\vartheta_1, \quad G = \frac{\mathfrak{M}\sin\epsilon}{\mathrm{r}^2}\cos\vartheta_2, \quad H = \frac{\mathfrak{M}\sin\epsilon}{\mathrm{r}^2}\cos\vartheta_3.$$

Accordingly, the magnitude of the vector potential of a magnetic particle at any point is $\frac{\mathfrak{M}\sin\epsilon}{r^2}$, and its direction is perpendicular to the axis of the magnet and the line joining its centre to the point.

If rotations from x to y, from y to z, and from z to x, be counter-clockwise, the rotation from the magnetic axis to radius vector is counter-clockwise as viewed from the positive end of the vector potential. The vector potential of a magnet of finite size is the resultant of the vector potentials of the magnetic particles of which it is composed.

195. **Magnetic Moment and Axis of Magnet.**—The potential energy of a magnet placed in a uniform field of force is determined from (9) by regarding α, β, γ, the components of force in the field, as constants; we have then

$$-W = \alpha\int A d\mathfrak{S} + \beta\int B d\mathfrak{S} + \gamma\int C d\mathfrak{S}. \qquad (19)$$

If we assume

$$\int A d\mathfrak{S} = Kl, \quad \int B d\mathfrak{S} = Km, \quad \int C d\mathfrak{S} = Kn, \quad l^2 + m^2 + n^2 = 1,$$

then l, m, n are the direction-cosines of a line, and we have

$$W = -KH\cos\theta, \qquad (20)$$

where H denotes the resultant uniform force, and θ the angle between its direction and that specified by l, m, n. This latter direction is fixed in the magnet, and the direction of H is fixed in space. Hence the magnet is acted on by a couple expressed by $-\frac{dW}{d\theta}$, that is, $-KH\sin\theta$, which tends to diminish θ and make the line l, m, n coincide with the direction of H.

Accordingly, the magnetic moment of the body, Art. 181, is expressed by K, and l, m, n are the direction-cosines of the magnetic axis.

If the potential of a magnet be expanded in a series of harmonics so that at an external point P, we have

$$V = \Sigma \frac{Y_i}{r^{i+1}},$$

where r denotes the distance of P from the origin, the first term $\frac{Y_0}{r}$ vanishes, since the total magnetic mass is zero, and in the second term, $\frac{Y_1}{r^2}$, the spherical harmonic Y_1 is $-K\cos\theta$. This is easily seen if we consider the expression for the potential energy W due to the presence of a mass at the point P. In this case W is given by the equation

$$W = mV = \frac{mY_1}{r^2} + \&c.$$

If we now suppose r to become infinite, but $\frac{m}{r^2}$ to be finite and equal to H, we have $W = HY_1$, but as the energy is that due to the presence of the magnet in a uniform field of force whose intensity is H, we have $W = -HK\cos\theta$.

Hence $$Y_1 = -K\cos\theta.$$

196. **Magnetic Shell.**—A magnetic shell may be defined as a *surface magnetized at each point in the direction of the normal.*

In this case, the expression for the magnetic moment of an element of the body is of the form $I\,d\nu\,dS$, where dS denotes an element of the surface, and $d\nu$ an element of its normal. The total magnetic moment of such a body is in general infinitely small; but if we suppose $Id\nu$ finite, this moment becomes finite. The quantity $Id\nu$ is termed the *strength of the magnetic shell*, and may be defined as the ratio of the magnetic moment of an element of the surface to its area. If we put $Id\nu = J$, then J denotes the strength of the magnetic shell.

When the strength of a magnetic shell is the same at all its points, J is constant, and the shell is said to be *uniform.*

197. **Potential of Uniform Magnetic Shell.**—If r denote the distance of an external point P from any point Q of the shell, by (28), Art. 54, the potential at P of the element of the shell at Q is $\dfrac{JdS}{r^2}\cos\epsilon$, where ϵ denotes the angle between r and the normal at Q.

But if $d\Omega$ denotes the solid angle which dS subtends at P, we have $r^2 d\Omega = dS\cos\epsilon$. Hence

$$\frac{JdS\cos\epsilon}{r^2} = Jd\Omega:$$

and the potential V of the shell at P is given by the equation

$$V = J\Omega, \qquad (21)$$

where Ω denotes the solid angle subtended by the shell at P.

This potential differs in character from those with which we have hitherto been concerned, as it is discontinuous at the surface of the shell.

If we regard as positive the side of the shell at which the north poles of the elements are situated, or towards which they point, the potential at the positive side exceeds that at the negative by $4\pi J$.

The solid angle subtended at P by the shell is in general the same as that subtended by its bounding curve, but the two solid angles differ in some important respects.

The solid angle subtended by the curve is continuous except at the curve itself, and in a circuit embracing the curve, by passing through its interior, is cyclic. Each time the circuit is completed the value of the solid angle is increased by 4π.

These characteristics of the two solid angles we shall now consider.

The solid angle subtended by the shell at P with its sign reversed is the same as Gauss's integral of the normal component of force emanating from a unit mass at P. The sign is reversed, because in Gauss's integral the positive direction of r is from P towards the surface; but, in the present case, the positive direction is from the surface towards P.

If P be on the positive side of the shell, the lines from P to the shell which fall inside the cone standing on the bounding curve meet the shell once externally, and possibly an even number of times afterwards. Those which fall outside this cone meet the shell twice, or some other even number of times: first, externally, and then internally, and therefore contribute nothing to the integral representing the solid angle. Accordingly, the two solid angles are the same when P is on the positive side of the shell, and when P is infinitely near the shell on this side, each may be denoted by Ω_1.

When P moves across the surface of the shell from the positive to the negative side, the solid angle subtended by the bounding curve remains unaltered, but that subtended by the shell becomes $\Omega_1 - 4\pi$. To see the truth of this we have only to suppose the closed surface completed of which the shell is part. Then, by Art. 26, the solid angle which the entire closed surface subtends at P is -4π; and it is plain that Ω_1 denotes the absolute magnitude of that part of this angle which is subtended by the portion of this surface which has been added to the shell. Hence the solid angle subtended by the shell is $-(4\pi - \Omega_1)$.

The solid angle subtended at P by the curve bounding the shell is everywhere continuous unless P be infinitely near the curve. As P moves about, the variations of the two solid

angles are the same except when P is passing through the surface of the shell. Hence we may take for the potential of the shell at P the expression $J\Omega$, where Ω denotes the solid angle subtended at P by the curve bounding the shell, with the proviso that when P passes through the shell from the positive to the negative side, $4J\pi$ must be subtracted from the foregoing expression.

If ψ be a function of the coordinates of a point, and if $\int \frac{d\psi}{ds} ds$ taken round a closed circuit be zero for every possible closed circuit, ψ is *acyclic*, but, if for some circuits $\int \frac{d\psi}{ds} ds$ taken round the circuit be not zero, ψ is *cyclic*. If a closed circuit s be such that we can draw a surface S, of which s is the boundary, so that at every point of S the function ψ has differential coefficients u, v, w which are finite and continuous, then by Stokes' Theorem, Art. 192, the function ψ must be acyclic for the circuit s. Again, if a surface fulfilling the conditions stated above be bounded by two curves, s_1 and s_2, the value of $\int \frac{d\psi}{ds} ds$ taken round the circuit is the same for one of these curves as for the other. It is now easy to see that Ω, the solid angle subtended at P by the curve s bounding the shell, is acyclic for every circuit which does not embrace this curve, passing through its interior. For since the differential coefficients of Ω are finite and continuous for all positions of P not infinitely near the curve s, this follows immediately from what has been said above.

If we suppose P to start from a point at an infinite distance on the positive side of the shell and to move in a straight line to a point at an infinite distance on the negative side, passing in its course through the interior of the curve s, the solid angle Ω passes from O to 4π. For if a unit sphere be described round P as centre, the edges of the cone having its vertex at P and standing on s initially converge to a point. As P approaches to s the cone opens out, and the area swept out on the sphere by the edges of the cone increases. After P passes through the interior of s, this area becomes greater than a hemisphere, and finally when P

reaches an infinite distance on the negative side of the shell, the edges of the cone again converge to a point on the sphere which is now opposite to that to which they originally converged. These edges have then swept out the entire sphere or 4π.

We may now suppose P to return to its original position along a path on the outside of s, and such that all its points are infinitely distant from s. At all these points the differential coefficients of Ω are zero; and hence the value of Ω is 4π, when P returns to its original position. It is now easy to see, from Stokes's theorem, that for any circuit passing through the interior of the curve and embracing it once

$$\int \frac{d\Omega}{ds}\, ds \quad \text{must be} \quad 4\pi.$$

Hence we conclude that the potential of a magnetic shell is expressed by a cyclic function, but that at the surface of the shell discontinuity occurs in the potential though not in the function. In consequence of this discontinuity the principle of the conservation of energy is maintained.

In fact, if P start from a point O on the surface of the shell, at the negative side, and travel round the edge of the shell till it reaches the point O' on the positive side of the shell, opposite and infinitely near to O, the function Ω increases by 4π, but in passing through the shell from O' to O the potential of the shell is diminished by 4π. Hence the value of the potential at O is unchanged by the motion of P round the complete circuit, but the value of Ω is increased by 4π.

198. **Energy due to Magnetic Shell.**—The energy due to a magnetic shell placed in an independent magnetic field is given by (9). If l, m, n denote the direction-cosines of the normal to the element dS of the shell, and J its strength, we have

$$A d\mathfrak{S} = J l dS, \quad B d\mathfrak{S} = J m dS, \quad C d\mathfrak{S} = J n dS,$$

and if α', β', γ' denote the components of magnetic force due to the field (9) becomes

$$W = -J\int (l\alpha' + m\beta' + n\gamma')\, dS. \qquad (22)$$

199. **Energy due to Two Magnetic Shells.**—If the magnetic field be due to a second shell S' whose components of force are $\alpha'\ \beta'\ \gamma'$, the energy W given by (22) represents the result of the mutual action of the two shells.

Since the one shell is outside the other, we may in (22) substitute the components of induction for those of force and for the components of induction we may put the expressions given by (13), Art. 193. Thus (22) becomes

$$W = -J\int\left\{l\left(\frac{dH'}{dy}-\frac{dG'}{dz}\right)+m\left(\frac{dF'}{dz}-\frac{dH'}{dx}\right)+n\left(\frac{dG'}{dx}-\frac{dF'}{dy}\right)\right\}dS$$

$$= -J\int\left(F'\frac{dx}{ds}+G'\frac{dy}{ds}+H'\frac{dz}{ds}\right)ds, \quad (23)$$

where s is the curve bounding the first shell.

The values of F', G', H', the components of the vector potential of the second shell, are given by (16), Art. 193. In this case

$$F' = \int\left\{B'\frac{d}{dz'}\left(\frac{1}{\mathrm{r}}\right)-C'\frac{d}{dy'}\left(\frac{1}{\mathrm{r}}\right)\right\}d\mathfrak{S}',$$

but $\quad B'd\mathfrak{S}' = J'm'dS', \quad C'd\mathfrak{S}' = J'n'dS',$

and hence $\quad F' = J'\int\left\{m'\frac{d}{dz'}\left(\frac{1}{\mathrm{r}}\right)-n'\frac{d}{dy'}\left(\frac{1}{\mathrm{r}}\right)\right\}dS'.$

In Stokes's theorem (12), Art. 192, if we make

$$u = \frac{1}{\mathrm{r}},\ v = 0,\ w = 0, \quad \text{we get} \quad F' = J'\int\frac{dx'}{ds'}\frac{1}{\mathrm{r}}\,ds',$$

where s' denotes the curve bounding the second shell. In a similar manner we have

$$G' = J'\int\frac{dy'}{ds'}\frac{1}{\mathrm{r}}\,ds', \quad H' = J'\int\frac{dz'}{ds'}\frac{1}{\mathrm{r}}\,ds'. \quad (24)$$

Substituting in (23) the values obtained for F', G', and H' we get

$$W = -JJ'\iint\left(\frac{dx}{ds}\frac{dx'}{ds'}+\frac{dy}{ds}\frac{dy'}{ds'}+\frac{dz}{ds}\frac{dz'}{ds'}\right)\frac{1}{\mathrm{r}}\,ds\,ds'$$

$$= -JJ'\iint\frac{\cos\varepsilon}{\mathrm{r}}\,ds\,ds', \quad (25)$$

where ε denotes the angle between the curve elements ds and ds'.

Section I.—*Induced Magnetism.*

200. **Magnetic Induction.**—When a body is placed in a field of magnetic force, in general its magnetism is altered. The magnetism produced by the force is called *induced magnetism.* When the magnetizing force is small, the induced magnetization is, in general, proportional to and codirectional with the total magnetic force acting at the point, so that if A_2 denote a component of induced magnetization, and a the corresponding component of the total magnetic force, $A_2 = \kappa a$, where κ is a coefficient depending on the nature of the body, and is called the coefficient of induced magnetization.

It is easy to see that A_2 and a are quantities of the same order, so that κ is a numerical, magnitude, which is positive in the case of paramagnetic bodies, and negative in the case of diamagnetic.

If A_1 denote the component of that part of the magnetization which is independent of induction, we have

$$A = A_1 + \kappa a, \quad B = B_1 + \kappa\beta, \quad C = C_1 + \kappa\gamma. \tag{1}$$

201. **Magnetism due altogether to Induction.**—If there be no magnetism in the body independent of the induction due to the field of force, $A_1 = B_1 = C_1 = 0$, and

$$A = \kappa a, \quad B = \kappa\beta, \quad C = \kappa\gamma. \tag{2}$$

In this case, by (3), Art. 183, we have

$$\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz} = \kappa\left(\frac{da}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}\right) = -4\pi\kappa\left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz}\right),$$

whence
$$\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz} = 0. \tag{3}$$

When the components of magnetization fulfil this condition, the distribution of magnetism is said to be *solenoidal,* and the potential corresponding may be regarded as due to a surface distribution of mass whose density is

$$lA + mB + nC.$$

202. **Components of Induction.**—When a body has no magnetism independent of that induced by the acting force, the components of induction are given by the equations

$$a = (1 + 4\pi\kappa)\alpha, \quad b = (1 + 4\pi\kappa)\beta, \quad c = (1 + 4\pi\kappa)\gamma. \quad (4)$$

If we put $1 + 4\pi\kappa = \varpi$, the quantity ϖ is called by Maxwell the specific magnetic inductive capacity, and by Thomson the magnetic permeability, and in the case of a body magnetically isotropic, having no permanent magnetism independent of induction, we have, then,

$$a = \varpi\alpha, \quad b = \varpi\beta, \quad c = \varpi\gamma. \quad (5)$$

203. **Distribution of Induced Magnetism.**—Let U denote the total magnetic potential, inside the body in which the distribution of induced magnetism is to be determined, U' the total potential in the external medium; then, as the distribution of induced magnetism is solenoidal, and there is no other magnetism inside the field in which U and U' are to be determined, we have

$$\nabla^2 U = 0, \quad \nabla^2 U' = 0,$$

also $U = U'$ at the surface bounding the magnetized body, and since, Art. 187, the normal component of induction is continuous,

$$\varpi \frac{dU}{d\nu} + \varpi' \frac{dU'}{d\nu'} = 0, \quad (6)$$

where ϖ and ϖ' denote the coefficients of permeability of the body and the external medium, and, ν and ν' the normals drawn into them at the separating surface.

If U' be assigned at the surface bounding the field externally, U and U' can be determined in only one way so as to satisfy the given conditions. Let us suppose that the equations could be satisfied by two pairs of functions U_1, U'_1 and U_2, U'_2, and let

$$\phi = U_1 - U_2, \quad \phi' = U_1' - U_2',$$

then, if S' be the surface bounding the field externally, we have

$$\int \phi \frac{d\phi}{d\nu} dS + \int \phi \nabla^2 \phi d\mathfrak{S} = -\int \left\{ \left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2 \right\} d\mathfrak{S},$$

$$\int \phi' \frac{d\phi'}{d\nu'} dS + \int \phi' \frac{d\phi'}{d\nu'} dS' + \int \phi' \nabla^2 \phi' d\mathfrak{S}' = -\int \left\{ \left(\frac{d\phi'}{dx}\right)^2 + \&c. \right\} d\mathfrak{S}'.$$

If we multiply the first of these equations by ϖ, the second by ϖ', and add, observing that at the surface S' we must have $\phi' = 0$, and that $\phi = \phi'$ at S, we get

$$\int \phi \left(\varpi \frac{d\phi}{d\nu} + \varpi' \frac{d\phi'}{d\nu'} \right) dS + \varpi \int \phi \nabla^2 \phi d\mathfrak{S} + \varpi' \int \phi' \nabla^2 \phi' d\mathfrak{S}'$$

$$= - \varpi \int \left\{ \left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2 \right\} d\mathfrak{S}$$

$$- \varpi' \int \left\{ \left(\frac{d\phi'}{dx}\right)^2 + \left(\frac{d\phi'}{dy}\right)^2 + \left(\frac{d\phi'}{dz}\right)^2 \right\} d\mathfrak{S}'.$$

Since $\varpi \frac{d\phi}{d\nu} + \varpi' \frac{d\phi'}{d\nu'} = 0$ at S, and $\nabla^2 \phi = 0$, $\nabla^2 \phi' = 0$,

the left-hand member of this equation is zero. The coefficients ϖ and ϖ' are always essentially positive, even if κ or κ' be negative. Hence each member of the right-hand side, and each of the terms under the integral signs, must vanish separately, and therefore $\phi' = 0$, $\phi = 0$.

204. **External Medium not Magnetic.**—If the external medium is not capable of being magnetized, we have $\kappa' = 0$, $\varpi' = 1$; also

$$U = V + \Omega, \quad U' = V + \Omega';$$

where V denotes the potential of the forces producing the induction, and Ω and Ω' the potentials, inside and outside the body, of the induced magnetism. In this case V is supposed to be given, and Ω' is zero at infinity.

205. **Anisotropic Medium.**—When a magnetic medium is anisotropic or crystalline, the induced magnetism is not, in general, codirectional with the magnetic force; but the components of induced magnetization are linear functions of the components of force, so that we have

$$\left.\begin{aligned} A &= \kappa_{11}\alpha + \kappa_{12}\beta + \kappa_{13}\gamma, \\ B &= \kappa_{21}\alpha + \kappa_{22}\beta + \kappa_{23}\gamma, \\ C &= \kappa_{31}\alpha + \kappa_{32}\beta + \kappa_{33}\gamma. \end{aligned}\right\} \tag{7}$$

By (9), Art. (189), we see that, to increase by $\delta\alpha$ the force acting in the element $d\mathfrak{S}$, the work required is $-A\delta\alpha$, and therefore we conclude that

$$-\frac{dW}{d\alpha} = A, \quad -\frac{dW}{d\beta} = B, \quad -\frac{dW}{d\gamma} = C. \tag{8}$$

Hence $\kappa_{21} = \kappa_{12}$, &c., and (7) become

$$\left.\begin{aligned} A &= \kappa_{11}\alpha + \kappa_{12}\beta + \kappa_{13}\gamma, \\ B &= \kappa_{12}\alpha + \kappa_{22}\beta + \kappa_{23}\gamma, \\ C &= \kappa_{13}\alpha + \kappa_{23}\beta + \kappa_{33}\gamma. \end{aligned}\right\} \tag{9}$$

206. **Isotropic Ellipsoid in Field of Uniform Force.**—If an ellipsoid, free from magnetism and surrounded by a non-magnetic medium, be placed in a field of uniform force, the distribution of induced magnetism can readily be determined. In fact, we may suppose the ellipsoid to be uniformly magnetized in a direction to be determined; and if the conditions of the question can thus be satisfied, we know by Art. 203 that we have reached the correct solution of the question.

Let I denote the intensity of the induced magnetization, and λ, μ, ν its direction-cosines; then, by (5), Art. 184, the potential V of the induced magnetism is given by the equation

$$V = I(\lambda Lx + \mu My + \nu Nz), \tag{10}$$

where, by (17), Art. 22, the constants L, M, N denote

$$2\pi abc\int_0^\infty \frac{du}{(a^2+u)^{\frac{3}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{1}{2}}},$$

the two other integrals obtained by interchanging b and c with a. Hence, if the components of the uniform force due to the field be denoted by F_1, F_2, F_3, the total magnetic force α, parallel to the axis of x, is $F_1 - I\lambda L$, and we have

$$I\lambda = \kappa\alpha = \kappa(F_1 - LI\lambda),$$

with two similar equations. Accordingly, we get

$$\left.\begin{aligned} (1+\kappa L)I\lambda &= \kappa F_1, \\ (1+\kappa M)I\mu &= \kappa F_2, \\ (1+\kappa N)I\nu &= \kappa F_3. \end{aligned}\right\} \qquad (11)$$

The values of I, λ, μ, and ν obtained from these equations satisfy the conditions of the question.

207. **Anisotropic Ellipsoid surrounded by Non-Magnetic Medium in Uniform Field of Force.**— In this case, if we proceed in a manner similar to that of the last Article, we get

$$I\lambda = \kappa_{11}(F_1 - I\lambda L) + \kappa_{12}(F_2 - I\mu M) + \kappa_{13}(F_3 - I\nu N),$$

and two similar equations; whence we have

$$\left.\begin{aligned} (1+\kappa_{11}L)I\lambda + \kappa_{12}MI\mu + \kappa_{13}NI\nu &= \kappa_{11}F_1 + \kappa_{12}F_2 + \kappa_{13}F_3, \\ \kappa_{12}LI\lambda + (1+\kappa_{22}M)I\mu + \kappa_{23}NI\nu &= \kappa_{12}F_1 + \kappa_{22}F_2 + \kappa_{23}F_3, \\ \kappa_{13}LI\lambda + \kappa_{23}MI\mu + (1+\kappa_{33})NI\nu &= \kappa_{13}F_1 + \kappa_{23}F_2 + \kappa_{33}F_3. \end{aligned}\right\} \qquad (12)$$

Hence I, λ, μ, and ν are determined.

SECTION III.—*Terrestrial Magnetism.*

208. **Earth's Magnetic Potential.**—The components of the Earth's magnetic force at any place can be determined by observation. This can be done either by finding the time of oscillation of a magnet, free to move in a horizontal or in a vertical plane, when disturbed from its position of equilibrium, or by arranging a position of equilibrium under the combined action of the Earth and magnets whose strength and position are known. The investigation of the Earth's magnetic potential was initiated by Gauss. In the British Islands some of the earliest observations were carried out by Lloyd in the magnetic observatories of Trinity College, Dublin.

When the Earth's horizontal force has been determined at a sufficient number of places, the question of the existence of an acyclic magnetic potential can be investigated.

If s denote any portion of a closed path on the Earth's surface, H the horizontal component of magnetic force at any point, and θ the angle which its direction makes with that of s, on the hypothesis that a magnetic potential V exists, we have $H \cos\theta = -\dfrac{dV}{ds}$. Hence, if an acyclic magnetic potential exists, $\int H \cos\theta \, ds$ taken round the closed path is zero. By finding a sufficient number of values of H and θ the numerical value of the integral can be determined approximately. In fact, if s_1 and s_2 correspond to stations not too far apart, we have

$$\int_{s_1}^{s_2} H \cos\theta \, ds = \tfrac{1}{2}(H_1 \cos\theta_1 + H_2 \cos\theta_2)(s_2 - s_1)$$

approximately. It is found in this way that $\int H \cos\theta \, ds$ taken round a closed path is always zero.

Hence we conclude that the magnetic action of the Earth can be represented by an acyclic potential, and consequently that electric currents passing from the outer atmosphere to the ground cannot be the cause of any part of this action.

209. **Locality of the Sources of the Earth's Magnetic Force.**—If the Earth's magnetic action be due to magnetism, or closed electric currents in its interior, the magnetic potential V at any point P outside its surface can be expanded in descending powers of r, the distance of P from the centre of the Earth. The difference between the numerical values of V at any two places can be determined from observations of the horizontal force. If the magnetic action be due to magnets or currents outside, the potential at any point nearer to the centre than the nearest of these sources of action can be expanded in ascending powers of r.

Hence for a point P close to the Earth's surface at its exterior we have

$$V = \Sigma U_i \frac{r^i}{a^{i+1}} + \Sigma Y_i \frac{a^i}{r^{i+1}}, \tag{1}$$

where a denotes the radius of the Earth and U_i and Y_i spherical harmonics. At the surface of the Earth

$$V = \frac{1}{a} \Sigma (U_i + Y_i),$$

and if a_i denote a coefficient in U_i, and b_i the coefficient of the corresponding term in Y_i, the coefficient of this term in V is $a_i + b_i$. By taking a sufficient number of numerical values of V at known places on the Earth's surface we can determine as many of these coefficients as we please so that we may regard $a_i + b_i$ as known.

If we now consider the vertical component Z, towards the centre, of the Earth's magnetic force, we have

$$Z = \frac{dV}{dr} = \Sigma i U_i \frac{r^{i-1}}{a^{i+1}} - \Sigma (i + 1) Y_i \frac{a^i}{r^{i+2}}. \tag{2}$$

At the surface (2) becomes

$$Z = \frac{1}{a^2} \Sigma (iU_i - (i + 1) Y_i). \tag{3}$$

Hence, from the observation of a sufficient number of values of Z we can determine

$$ia_i - (i + 1)\, b_i,$$

and consequently a_i and b_i are each known.

It is found that a_i is always zero, and accordingly we conclude that the Earth's magnetic action is due altogether to sources inside its surface, and that V, the potential of the Earth's magnetic action, is given by the equation

$$V = \Sigma Y_i \frac{a^i}{r^{i+1}}. \tag{4}$$

210. **Earth's Magnetic Poles.**—A magnetic pole is a point at which the horizontal force vanishes. At such a point this force changes sign so that at each side of the pole the same end of the needle points towards the pole.

If there be two poles of the same kind on the Earth's surface in going from one to the other along a magnetic meridian, the horizontal force must change sign and therefore vanish. Hence there must be a third pole between the two former. The end of the needle which pointed towards these poles points away from the intermediate one at each side.

As a matter of fact there are only two magnetic poles on the Earth's surface, and these two are of opposite kinds. The proximity of these poles to the extremities of the Earth's axis of rotation appears to indicate a connexion between the Earth's magnetism and the Earth's rotation. From the properties of electric currents it is easy to see that such currents circulating round the Earth, and approximately parallel to the equator, would account for the magnetic phenomena exhibited.

CHAPTER X.

ELECTRIC CURRENTS.

211. **Introductory.**—Not long after the discovery of current electricity it was observed by Oersted that a wire through which an electric current is passing exercises an attractive or repulsive force upon the pole of a magnet-needle. It was found also that wires along which electric currents are passing attract or repel one another.

By a combination of experimental and mathematical investigations Ampère succeeded in arriving at the laws which regulate the attraction of currents on each other and on magnets.

His original investigations must ever be regarded as worthy of the highest admiration, but some of his experiments, combined with the theoretical developments of other physicists, enable us to arrive at his results by methods shorter and simpler than those employed by him.

212. **Electric Currents.**—An electric current may be produced in various ways; but in all cases the maintenance of an electric current requires an expenditure of energy supplied by an external source.

The source of energy may be chemical, as when two substances unite chemically, or mechanical, such as the action of a steam-engine or water-mill, but in all cases there must be a source of energy outside the current itself on which its continuance depends. The force due to an electric current is not therefore a permanent natural force, and propositions depending on the principle of energy cannot be applied to it in the same manner as to gravitation, or to the attraction of static electricity.

The currents whose attraction we are about to consider are those transmitted along a wire of small section.

The quantity of electricity which passes through an orthogonal section of the wire in the unit of time is called the *strength of the current.* The quantity which passes through the unit of area is called its *intensity.* When a steady current is established, the strength is uniform throughout the wire. The force which causes and keeps up the current is the electric force. When there is a potential corresponding to this force, the force is the rate of diminution of the potential, or $-\frac{dV}{ds}$, where s denotes an element of length along the wire. As the current is supposed to be constant, this force must be equilibrated by another of equal magnitude.

The current is thus analogous to the uniform motion of a body sliding on a rough surface.

The retarding force on a unit of electricity is found to be proportional to the intensity of the current, that is, its strength per unit of area.

Thus we have

$$-\frac{dV}{ds} = k\frac{i}{\sigma},$$

where k is a coefficient depending on the material of the wire, σ denotes the area of its section, and i the strength of the current. If we integrate the equation above, we get

$$V_1 - V_2 = i\int\frac{k}{\sigma}\,ds.$$

If k and σ be constant, this becomes

$$V_1 - V_2 = \frac{k}{\sigma}li = Ri, \qquad (1)$$

where l denotes the length of the wire, and $R = \frac{kl}{\sigma}$ ...ry. ...ne size

quantity R is termed the resistance of the wire.

If $V_1 - V_2$, the difference between the v...tric circuit potential at the extremities of the wire, be ...quivalent to a equation (1) may be written ...ar to dS and

$$E = Ri.$$

...e strength of the

This expresses what is called Ohm's Law. E is termed the *electromotive force*, and may be defined as *the difference in potential between the extremities of the wire*, or, more generally, as *the line integral taken along the wire of the electromotive intensity*.

The term 'electromotive force' applied to this integral seems highly objectionable, but is sanctioned by long usage.

213. **Solenoids.**—If a wire be bent into the form of a circle, not quite closed, be carried on for a short distance at right angles to the plane of the circle, bent into another circle equal and parallel to the first, carried on again, and so on, and finally brought back in a straight line perpendicular to the planes of the circles and close to the connecting portions of wire between them; and if an electric current be sent through the wire, we obtain what is termed a *solenoid*. As the portion of the current which is perpendicular to the planes of the circles consists of two parallel parts close together and flowing in opposite directions, it produces no attraction on a magnet-pole, and the solenoid may be regarded as being composed of a number of equal circles whose planes are perpendicular to a straight line passing through their centres.

It is found that at distances which are large compared with the diameter of one of the circles, the solenoid exercises the same action as a linear magnet.

If σ denote the area of one of the circles, δ the perpendicular distance between two of them, l the length of the solenoid, and i the strength of the current, it is found that the magnetic moment of the solenoid is expressed by

$$\frac{i\sigma}{\delta}\,l.$$

muThe magnetic moment of a linear magnet of equal length, which of small magnets having each a magnetic moment μ current is positions de length h, is expressed by $\frac{\mu}{h}l$. The axial length h applied to it between the centres of two of the small magnets attraction of s... magnet is composed. If the ... the moment of

The currents ... that of the ... are ... magnet, and if we suppose are those transmitted ... wire ...

As the equivalence of the solenoid to the magnet holds good, whatever be the number of circles in the solenoid, we conclude that,

A small circular current is equivalent to a small magnet whose centre coincides with that of the circle, whose axis is perpendicular to the plane of the circle, and whose moment is equal to the area of the circle multiplied by the strength of the current.

The equivalence of a solenoid to a linear magnet holds good equally well if another plane curve be substituted for a circle, and becomes more rigorously true according as the diameter of the curve is diminished, compared with the distance of the magnet on which the solenoid acts. Hence we conclude that,

The magnetic action of an infinitely small electric circuit is equivalent to that of a magnetic particle whose axis is surrounded by the circuit and is perpendicular to its plane, and whose magnetic moment is equal to the area of the circuit multiplied by the strength of the current.

214. **Equivalence of Electric Circuit to Magnetic Shell.**—If a single-sheeted surface be described of which an electric circuit is the boundary, and a network of lines be drawn on this surface dividing it into a number of small elements, the electric current is equivalent to a current of equal strength circulating in its direction (clockwise or counter-clockwise) round each of these elements. This is obvious if we remember that along the boundary line between two adjacent elements there are two currents in opposite directions, one for each element. As these currents are equal, they neutralize each other; and the only current which remains uncompensated is that in the outer boundary. By increasing the number of lines in the network, the size of each element can be diminished without limit.

From Art. 213 it appears that the electric circuit embracing the element dS of the surface is equivalent to a magnetic particle whose axis is perpendicular to dS and whose moment is idS, where i denotes the strength of the current.

Hence, the total electric circuit is equivalent to the assemblage of small magnets, normal to the surface S, whose moments are the areas of the elements surrounding the magnets multiplied by the strength of the current, that is, to the magnetic shell whose surface is S and whose strength is i.

215. **Magnetic Potential of Electric Circuit.**—Since the magnetic action of an electric circuit is the same as that of a magnetic shell bounded by the circuit, the magnetic potential of an electric circuit whose strength is i at a point P is expressed by $i\Omega$, where Ω denotes the solid angle subtended by the circuit at P. This potential is continuous everywhere except at the circuit itself.

For any closed curve not passing through the space surrounded by the circuit the potential is acyclic.

For a curve passing through this space and embracing the circuit the potential is cyclic, and the value of the cyclic constant is $4\pi i$.

These characteristics of the potential show that in moving a magnet-pole round a closed curve which does not embrace the circuit no work is done, but that in moving the unit pole round a curve embracing the circuit and passing through its interior, if the direction of motion be opposed to the force, work is done represented by $4\pi i$.

If we imagine a person to stand on the positive side of a shell equivalent to the current, that is, on the side towards which the north poles point, the current as seen by him will circulate counter-clockwise, and if a person is placed lying along the current which enters at his feet and goes out at his head, the motion of a north magnetic pole moved by the current round his body will as seen by him be counter-clockwise.

The first of these statements is deducible from the experiments made on solenoids; the second follows from the equivalence of the current to the magnetic shell.

216. **Magnetic Force of Currents.**—Since an electric circuit is equivalent to a magnetic shell, the components of force due to the current are in space outside the shell the same as α, β, γ, the components of magnetic force due to the shell.

Outside the shell α, β, γ are the same as a, b, c, the components of induction due to the shell. At the shell α, β, γ are discontinuous, Art. 186; but since the magnetization of the shell is normal to its surface, a, b, c are continuous, Art. 187. The force-components of the current are everywhere continuous except at the current itself. Hence we conclude that for all space outside the current, the components of its magnetic force are expressed by a, b, c, being the same as the components of induction of the equivalent magnetic shell.

217. **Energy due to presence of Electric Current in Independent Magnetic Field.**—Let α', β', γ' denote the components of magnetic force; a', b', c' those of induction, due to a magnetic shell S equivalent to the current; α, β, γ the components of magnetic force; a, b, c those of induction due to the field $\mathfrak{S}$, and A, B, C the components of its magnetization. Let U denote the energy due to the presence of the shell in the field, and W that due to the presence of the current.

By Art. 216 and (9), Art. 189, we have

$$U = -\int (\alpha' A + \beta' B + \gamma' C)\, d\mathfrak{S},$$

$$W = -\int (a' A + b' B + c' C)\, d\mathfrak{S}.$$

Except at the surface of the shell, $a' = \alpha'$, $b' = \beta'$, $c' = \gamma'$; but at S we have $a' d\mathfrak{S} = \alpha' d\mathfrak{S} + 4\pi i l dS$, where l denotes the direction-cosine of the normal to S, with similar equations for b' and c'.

Hence $$W = U - 4\pi i \int (lA + mB + nC)\, dS.$$

Again, by (22), Art. 198,

$$U = -i \int (l\alpha + m\beta + n\gamma)\, dS,$$

and therefore

$$W = -i \int \{l\alpha + m\beta + n\gamma + 4\pi (lA + mB + nC)\} dS$$

$$= -i \int (la + mb + nc)\, dS. \qquad (3)$$

218. **Force-Components of Current expressed as Integrals.**—If a, b, c denote the components of magnetic induction due to a shell equivalent to the current, by Art. 193, and (24), Art. 199, we have

$$a = \frac{dH}{dy} - \frac{dG}{dz} = \frac{d}{dy}\int\frac{idz'}{\mathrm{r}} - \frac{d}{dz}\int\frac{idy'}{\mathrm{r}},$$

where x', y', z' denote the coordinates of a point on the circuit, and r the distance between this point and the point x, y, z, and the integrals are taken round the entire circuit.

Since
$$\frac{d}{dy}\frac{1}{\mathrm{r}} = -\frac{1}{\mathrm{r}^2}\,\frac{y-y'}{\mathrm{r}},$$

we get
$$a = i\int\left(\frac{dy'}{ds'}\,\frac{z-z'}{\mathrm{r}} - \frac{dz'}{ds'}\,\frac{y-y'}{\mathrm{r}}\right)\frac{ds'}{\mathrm{r}^2},$$

with similar expressions for b and c; and if F_1, F_2, F_3 denote the components of force exercised by a circuit of strength i on a magnet-pole of strength m, situated at the point x, y, z, we have

$$\left.\begin{aligned} F_1 &= im\int\left(\frac{dy'}{ds'}\,\frac{z-z'}{\mathrm{r}} - \frac{dz'}{ds'}\,\frac{y-y'}{\mathrm{r}}\right)\frac{ds'}{\mathrm{r}^2},\\ F_2 &= im\int\left(\frac{dz'}{ds'}\,\frac{x-x'}{\mathrm{r}} - \frac{dx'}{ds'}\,\frac{z-z'}{\mathrm{r}}\right)\frac{ds'}{\mathrm{r}^2},\\ F_3 &= im\int\left(\frac{dx'}{ds'}\,\frac{y-y'}{\mathrm{r}} - \frac{dy'}{ds'}\,\frac{x-x'}{\mathrm{r}}\right)\frac{ds'}{\mathrm{r}^2}. \end{aligned}\right\} \quad (4)$$

219. **Force exerted by Element of Current on Magnet-Pole.**—The components of force given by (4) are the sums of the components of force contributed by the various elements of the circuit.

Hence, the circuit acts as if the force-components due to a single element ds' of a current whose strength is i, acting on a magnet-pole of strength m, were expressed by

$$\frac{imds'}{\mathrm{r}^2}\left(\frac{dy'}{ds'}\,\frac{z-z'}{\mathrm{r}} - \frac{dz'}{ds'}\,\frac{y-y'}{\mathrm{r}}\right),$$

$$\frac{imds'}{\mathrm{r}^2}\left(\frac{dz'}{ds'}\,\frac{x-x'}{\mathrm{r}} - \frac{dx'}{ds}\,\frac{z-z'}{\mathrm{r}}\right),$$

$$\frac{imds'}{\mathrm{r}^2}\left(\frac{dx'}{ds'}\,\frac{y-y'}{\mathrm{r}} - \frac{dy'}{ds'}\,\frac{x-x'}{\mathrm{r}}\right).$$

That these are the actual force-components due to a current element is shown at the end of this Article.

In the above equations,

$$\frac{dx'}{ds'}, \quad \frac{dy'}{ds'}, \quad \text{and} \quad \frac{dz'}{ds'}$$

are the direction-cosines of the current element ds', and

$$\frac{x - x'}{\mathrm{r}}, \quad \frac{y - y'}{\mathrm{r}}, \quad \text{and} \quad \frac{z - z'}{\mathrm{r}}$$

those of r. Hence, if θ denote the angle between ds' and r, and ϑ_1, ϑ_2, ϑ_3 the direction-angles of a perpendicular to their plane, the force-components due to the current element are expressed by

$$\frac{im \sin\theta ds'}{\mathrm{r}^2}\cos\vartheta_1, \quad \frac{im \sin\theta ds'}{\mathrm{r}^2}\cos\vartheta_2, \quad \frac{im \sin\theta ds'}{\mathrm{r}^2}\cos\vartheta_3.$$

Hence the force which a current element ds' of strength i exerts on a magnet-pole of strength m is perpendicular to the plane containing the pole and the current element, and tends to make the pole move in a counter-clockwise direction round the current element, along which the observer is supposed to be situated with the current entering at his feet and going out at his head. The magnitude of the force is

$$\frac{im \sin\theta ds'}{\mathrm{r}^2}.$$

This result can be proved directly from the expression for the magnetic potential of the circuit.

If we suppose an element ds' of the circuit to be free to undergo a displacement under the action of a magnetic pole m, the work done by the force in this displacement will be equal to the loss of potential energy of the system.

The potential energy W of the system is denoted by $im\Omega$, where Ω is the solid angle subtended at m by a surface S bounded by the circuit.

Let ds' receive three displacements: one, $\delta\xi$, along ds' itself; one, $\delta\eta$, perpendicular to ds' in the plane of ds' and r; and one, $d\zeta$, perpendicular to the two former. $\delta\xi$ does not alter the surface S. The displacement $\delta\eta$ by the motion of ds'

generates an increment of the surface S, but the element of surface so generated is in a plane containing r; and, as its normal is perpendicular to r, it subtends no solid angle at m. The displacement $\delta\zeta$ alters S by the amount $ds'\delta\zeta$, and the normal to this element of surface lies in the plane of r and ds', and is perpendicular to the latter. Hence, if θ denote the angle between ds' and r, the angle between r and the normal is $\frac{\pi}{2} - \theta$. Accordingly, the element of solid angle subtended at m by the element of surface is

$$\frac{\sin\theta\, ds'\,\delta\zeta}{\mathrm{r}^2},$$

and therefore $$\delta W = im\,\delta\Omega = \frac{im\sin\theta\, ds'}{\mathrm{r}^2}\,\delta\zeta.$$

Hence the force between m and ds' is in the direction of the displacement $\delta\zeta$, and is expressed by

$$\frac{im\sin\theta\, ds'}{\mathrm{r}^2}.$$

The direction in which the force exerted by ds' on m tends to move the latter is in the direction in which the solid angle Ω at m is diminishing. Thus, we arrive at the results already stated.

220. **Energy due to mutual action of two Electric Circuits.**—Since the action of each circuit in space outside itself is the same as that of a magnetic shell, if W denote the energy due to the mutual action, by (25), Art. 199, we have

$$W = -ii'\int\frac{\cos\varepsilon}{\mathrm{r}}\,ds\,ds'. \tag{5}$$

It is here assumed that the strength of each current is maintained constant.

221. **Forces between two Electric Circuits.**—If X, Y, Z denote the components of the force acting on a current element in consequence of the mutual action between the circuits, for any system of small displacements we have

$$\Sigma\,(X\delta x + Y\delta y + Z\delta z) = -\,\delta W.$$

In order to determine the variation of W we must express $\cos\varepsilon$ in terms of r and its differential coefficients.

If x, y, z denote the coordinates of an element, ds of one current, and x', y', z' those of an element, ds' of the other, and r the distance between these elements, remembering that x, y, z are functions of s, and x', y', z' of s', and that s and s' are independent of each other, we have

$$r^2 = (x - x')^2 + (y - y')^2 + (z - z')^2,$$

$$r\frac{dr}{ds} = (x - x')\frac{dx}{ds} + (y - y')\frac{dy}{ds} + (z - z')\frac{dz}{ds},$$

$$\frac{dr}{ds}\frac{dr}{ds'} + r\frac{d^2r}{ds\,ds'} = -\left(\frac{dx}{ds}\frac{dx'}{ds'} + \frac{dy}{ds}\frac{dy'}{ds'} + \frac{dz}{ds}\frac{dz'}{ds'}\right) = -\cos\varepsilon.$$

Substituting for $\cos\varepsilon$ in (5) we get

$$W = i\,i'\iint\left(\frac{d^2r}{ds\,ds'} + \frac{1}{r}\frac{dr}{ds}\frac{dr}{ds'}\right)ds\,ds'.$$

The first term under the integral sign can be integrated round either circuit, and, as the circuits are closed, it vanishes.

Hence

$$\delta W = i\,i'\iint\left\{\frac{1}{r}\left(\frac{dr}{ds}\frac{d\delta r}{ds'} + \frac{dr}{ds'}\frac{d\delta r}{ds}\right) - \frac{dr}{ds}\frac{dr}{ds'}\frac{\delta r}{r^2}\right\}ds\,ds'.$$

If we integrate by parts the first two terms, since the circuits are closed, the single integrals vanish, and we have

$$\begin{aligned}
\delta W &= -i\,i'\iint\left\{\frac{1}{r^2}\frac{dr}{ds}\frac{dr}{ds'} + \frac{d}{ds}\left(\frac{1}{r}\frac{dr}{ds'}\right) + \frac{d}{ds'}\left(\frac{1}{r}\frac{dr}{ds}\right)\right\}\delta r\,ds\,ds' \\
&= i\,i'\iint\left\{-\frac{2}{r}\frac{d^2r}{ds\,ds'} + \frac{1}{r^2}\frac{dr}{ds}\frac{dr}{ds'}\right\}\delta r\,ds\,ds' \\
&= i\,i'\iint\left\{\frac{2\cos\varepsilon}{r^2} + \frac{3}{r^2}\frac{dr}{ds}\frac{dr}{ds'}\right\}\delta r\,ds\,ds' \\
&= i\,i'\iint\frac{2}{r^2}\left\{\cos\varepsilon + \frac{3}{2}\cos\theta\cos\theta'\right\}\delta r\,ds\,ds', \qquad (6)
\end{aligned}$$

where θ and θ' denote the angles which r makes with s and s'.

Hence

$$\Sigma(X\delta x + Y\delta y + Z\delta z)$$

$$= -ii'\iint \frac{2}{\mathrm{r}^2}\left(\cos\epsilon + \frac{3}{2}\cos\theta\cos\theta'\right) ds\,ds'\,\delta\mathrm{r}. \qquad (7)$$

Accordingly, the forces due to the mutual action of the circuits are equivalent to a system of forces acting in the lines joining the elements of one circuit to those of the other. If R denote the magnitude of the force acting in the line joining the elements ds and ds', by (7), we have

$$R = -\frac{2ii'}{\mathrm{r}^2}\left(\cos\epsilon + \frac{3}{2}\cos\theta\cos\theta'\right) ds\,ds'. \qquad (8)$$

The negative sign shows that the force between the elements is attractive when the currents are both approaching the shortest distance between their lines of direction.

The magnitude of R was discovered by Ampère. He assumed that the direction of the force between two current elements is the line joining them.

In the investigation above, nothing has been assumed; but it has been shown that two closed currents act on each other as if there were a force R along each line joining an element of one current to an element of the other.

So far as this investigation goes there may be other forces acting between each pair of elements, but these forces must be such as to produce no effect on the total action between two closed currents.

If U_1, U_2, and U_3 denote three functions of s and s', in addition to R acting along r, there might be three forces:

$\frac{dU_1}{ds'}ds\,ds'$ parallel to the axis of x,

$\frac{dU_2}{ds'}ds\,ds'$ parallel to that of y, and

$\frac{dU_3}{ds'}ds\,ds'$ parallel to that of z,

due to the action of ds' on ds.

In this c. ...
on ds, resulti

taken round the closed circuit s, and ...

As the expression for the force between two ele... must be symmetrical with respect to these elements, t... force exercised by ds on ds' parallel to the axis of x would, in this case, be

$$\frac{dU_1}{ds}\,ds\,ds';$$

and as this must be equal and opposite to the force exercised by ds' on ds, we have

$$\frac{dU_1}{ds} = -\frac{dU_1}{ds'}.$$

Again, as U is a function of s and s',

$$dU_1 = \frac{dU_1}{ds}\,ds + \frac{dU_1}{ds'}\,ds' = \frac{dU_1}{ds}\,(ds - ds').$$

Hence $\dfrac{dU_1}{ds}$ is a function of $s - s'$, and therefore

$$U_1 = f_1(s - s').$$

In like manner,

$$U_2 = f_2(s - s'), \qquad U_3 = f_3(s - s').$$

222. **Force on Current Element in Magnetic Field.**—If λ, μ, ν denote the direction-cosines of a current element ds, we have seen, Art. 219, that the components of the force which a magnet-pole exerts on ds are

$$(\mu\gamma - \nu\beta)\,ids, \quad (\nu\alpha - \lambda\gamma)\,ids, \quad \text{and} \quad (\lambda\beta - \mu\alpha)\,ids,$$

where α, β, γ denote the components of the magnetic force due to the magnet-pole.

225. **Electric Displacement or Polarization.**—When a conductor is electrically excited the conductors in the vicinity become electrically excited also, and a change is produced in the intervening medium or dielectric whereby at each point a directed or vector quantity is brought into existence in the medium.

This directed quantity is called by Maxwell the *electric displacement*, and by Professor J. J. Thomson the *electric polarization*. The latter term is no doubt scientifically the more correct; but the word 'polarization' is used so frequently, especially in the theory of light, that Maxwell's term is in practice the more convenient.

In order to bring about this change in the dielectric the expenditure of work is required. If the electric displacement per unit of volume be denoted by D, and its components by f, g, h, the expression for the total work δU per unit of volume, required to increase D by δD, is of the form

$$X\delta f + Y\delta g + Z\delta h.$$

The quantities by which δf, δg, and δh are multiplied in this expression are called the components of the electromotive intensity R.

It will be shown that the vector quantity thus defined has properties for the most part the same as those which belong to the electromotive intensity in the theory of action at a distance.

Since $X\delta f dx dy dz$ represents an element of work, $X\delta f dx dy$ is of the nature of a mechanical force. Hence, if X be regarded as of the same nature as the force acting on the unit of electricity, $f dx dy$ may be regarded as a quantity of electricity, and f as a surface-density.

In an isotropic dielectric whose properties are the same in every direction, *the electromotive intensity is co-directional with, and proportional to, the electric displacement.* Hence for such a dielectric we may write

$$4\pi f = kX, \quad 4\pi g = kY, \quad 4\pi h = kZ. \tag{1}$$

The constant k depends on the nature of the dielectric and is called its specific inductive capacity.

Since f is of the nature of an electric surface-density, by (5), Art. 29, k must be a numerical quantity.

226. **Energy due to Electric Displacement.**—If U denote the energy per unit of volume due to an electric displacement, by Art. 224, we have

$$\delta U = X\delta f + Y\delta g + Z\delta h.$$

Substituting for X, Y, Z, from (1) we get, by integration,

$$U = \frac{2\pi}{k}\left(f^2 + g^2 + h^2\right) = \frac{k}{8\pi}\left(X^2 + Y^2 + Z^2\right)$$
$$= \tfrac{1}{2}(Xf + Yg + Zh). \qquad (2)$$

Hence the total energy W, stored up in an isotropic dielectric $\mathfrak{S}$ in consequence of an electric displacement, is given by the equations

$$W = \frac{2\pi}{k}\int D^2 d\mathfrak{S} = \frac{k}{8\pi}\int R^2 d\mathfrak{S} = \tfrac{1}{2}\int RD\,d\mathfrak{S}. \qquad (3)$$

The second of the expressions for W given by (3) differs from that in Art. 77 only by containing the factor k.

227. **Conductors and Currents.**—A permanent electric displacement cannot be set up in a conductor, but passes away immediately if not renewed. A displacement which is continually passing on and being continually renewed constitutes an electric current. The intensity of a current is the *rate of change of the corresponding displacement.* When a conductor in electric equilibrium is situated in a dielectric in which there is a displacement, it constitutes a boundary to the dielectric; and the surface integral of the normal component of the displacement taken over the conductor constitutes what is called *the charge on the conductor.*

228. **Solenoidal Distribution of Displacement.**—If a closed curve be drawn in a dielectric, and through each of its points a line be drawn in the direction of the electric displacement, we have what is called a tube of induction, or, in the language of Professor J. J. Thomson, a Faraday tube. Such a tube terminates at each end on a conductor, and, whatever be the electric charge at one end, an equal and opposite charge is found at the other. In an isotropic medium

tubes of induction are in the same direction as tubes of force, and are therefore at right angles to the surface of a conductor in electric equilibrium. Hence, if the tube be small, the positive displacement over the normal section directed into the tube at one end is equal in magnitude to the negative displacement directed into the tube over the normal section at the other end. Hence if Σ_1 and Σ_2 denote the two normal sections, and D_1 and D_2 the two displacements in the positive direction of the line of induction, we have $D_1\Sigma_1 = D_2\Sigma_2$.

We conclude that, for any small tube of induction drawn in the dielectric, the product of the displacement and the normal section is constant.

From this it follows that, if any closed surface S be drawn whose interior is occupied continuously by the dielectric, and if l, m, n denote the direction-cosines of the normal, we have

$$\int (lf + mg + nh)\, dS = 0.$$

For, if ψ be the angle which a line of induction makes with the normal to the surface at any point,

$$D\Sigma = D \cos\psi\, dS = (lf + mg + nh)\, dS;$$

and, as every tube of induction is cut twice, or some other even number of times by the closed surface,

$$\int (lf + mg + nh)\, dS = \int D \cos\psi\, dS = 0. \tag{4}$$

If the volume enclosed by S be the element $dx\, dy\, dz$, we obtain

$$f\, dy\, dz - \left(f + \frac{df}{dx}\, dx\right) dy\, dz + g\, dz\, dx - \left(g + \frac{dg}{dy}\, dy\right) dz\, dx$$
$$+ h\, dx\, dy - \left(h + \frac{dh}{dz}\, dz\right) dx\, dy = 0;$$

that is,
$$\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0. \tag{5}$$

This equation expresses a fundamental property of the electric displacement, and is analogous to the condition fulfilled by the components of velocity in an incompressible fluid.

In the case of a conductor, f, g, h cannot exist except in the form

$$\frac{df}{dt}\,dt, \text{ \&c.};$$

but the solenoidal condition is still fulfilled, so that for a conductor we have

$$\frac{d}{dt}\left(\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}\right)=0. \tag{6}$$

229. **Constancy of Charge on Insulated Conductor.**—If a conductor be insulated, its bounding surface, or surfaces, remains unchanged, and throughout the conductor by (6) we have

$$\frac{d}{dx}\frac{df}{dt}+\frac{d}{dy}\frac{dg}{dt}+\frac{d}{dz}\frac{dh}{dt}=0.$$

Multiplying by $dx\,dy\,dz$, and integrating, throughout the conductor we get

$$\int\left(l\frac{df}{dt}+m\frac{dg}{dt}+n\frac{dh}{dt}\right)dS=0;$$

that is,
$$\frac{d}{dt}\int\Big(lf+mg+nh\Big)\,dS=0.$$

Hence $\int(lf+mg+nh)\,dS$, taken over the surface or surfaces of the conductor, is constant.

When a conductor is touched by another conductor, the bounding surface of the space through which the integration is effected is altered, and there is no longer any ground for asserting the constancy of the charge.

230. **Displacement due to Electrified Sphere.**—If a conducting sphere, of radius a, placed in an isotropic medium, be uniformly electrified, the lines of force and of induction are perpendicular to its surface and pass through its centre, since there is perfect symmetry round this point.

Hence the sphere is in electric equilibrium, and over any concentric sphere of radius r the displacement D is uniformly distributed; and if D_0 denote the displacement at the surface of the sphere of radius a, we have $4\pi r^2 D = 4\pi a^2 D_0 = e$, where e denotes the total charge on the electrified sphere. Hence

$$D = \frac{e}{4\pi r^2}.$$

If a be sufficiently small, we may regard the sphere as an electrified particle.

The electromotive intensity R is given by the equation

$$R = \frac{4\pi}{k} D = \frac{1}{k} \frac{e}{r^2}; \qquad (7)$$

and we have the result, that in an isotropic medium the force due to an electrified particle varies directly as the charge on the particle and inversely as the square of the distance.

231. **Energy due to two Small Electrified Spheres.**—Let the radii of the spheres be denoted by a and β, and the spheres themselves by A and B. The electromotive intensity due to the sphere A, on which there is a charge e_1, is by Art. 229, on the hypothesis that the charge is uniformly distributed, $\frac{e_1}{kr_1^2}$, where r_1 denotes the distance from the centre of the sphere. The electromotive intensity due to the sphere B is in like manner $\frac{e_2}{kr_2^2}$.

It is plain that the resultant force may be derived from a potential function V, where

$$kV = \frac{e_1}{r_1} + \frac{e_2}{r_2} = k(V_1 + V_2).$$

If W be the energy due to the spheres, we have, then,

$$8\pi W = k\int R^2 d\varpi = k\int\left\{\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right\} d\varpi$$

$$= k\left\{-\int V\frac{dV}{d\nu}dS - \int V\nabla^2 V d\varpi\right\}.$$

The surface integral is to be taken over a sphere of infinite radius and over the spheres whose radii are α and β.

At the surface S_1 of the sphere A, if α be sufficiently small,

$$kV = \frac{e_1}{\alpha} + \frac{e_2}{c},$$

where c denotes the distance between the centres of A and B, and

$$k\frac{dV}{d\nu} = -\frac{e_1}{\alpha^2} + k\frac{dV_2}{d\nu}.$$

By Art. 26,
$$\int \frac{dV_2}{d\nu}\,dS_1 = 0,$$

and, as V is constant at S_1, we have

$$\int V\frac{dV_2}{d\nu}\,dS_1 = 0\,;$$

Hence
$$\int V\frac{dV}{d\nu}\,dS_1 = -\frac{4\pi}{k^2}\left(\frac{e_1}{\alpha} + \frac{e_2}{c}\right)\frac{e_1}{\alpha^2}\,\alpha^2.$$

In like manner,

$$\int V\frac{dV}{d\nu}\,dS_2 = -\frac{4\pi}{k^2}\left(\frac{e_2^2}{\beta} + \frac{e_1 e_2}{c}\right).$$

The integral over the sphere of infinite radius is zero, also $\nabla^2 V = 0$ throughout the field. Hence

$$8\pi W = \frac{4\pi}{k}\left(\frac{e_1^2}{\alpha} + \frac{e_2^2}{\beta} + \frac{2e_1e_2}{c}\right),$$

and
$$W = \frac{1}{2k}\left(\frac{e_1^2}{\alpha} + \frac{e_2^2}{\beta} + \frac{2e_1e_2}{c}\right). \qquad (8)$$

If the sphere A were alone in the field, the expression above would become $\dfrac{e_1^2}{2k\alpha}\cdot$ Similarly, if B were alone, it would be $\dfrac{e_2^2}{2k\beta}\cdot$ Hence the energy due to the mutual action of the two spheres is $\dfrac{e_1e_2}{kc}\cdot$

232. **Force between Electrified Particles.**—If W denote the energy due to the mutual action of two electrified particles, by Art. 231 we have

$$W = \frac{e_1 e_2}{kr},$$

where r denotes the distance between them. Hence, if F be the mutual force which they exercise on one another,

$$F = -\frac{dW}{dr} = \frac{1}{k}\frac{e_1 e_2}{r^2}. \qquad (9)$$

Accordingly, the force between two electric particles acts in the line between them, and varies directly as the product of the quantities of electricity and inversely as the square of the distance.

Also, by (7), Art. 230, the electromotive intensity due to an electric particle is equal to the force which it exercises on the unit of electricity.

233. **Irrotational Distribution of Electromotive Intensity.**—The components X, Y, Z of the electromotive intensity, due to a permanent statical distribution of electricity, must be the differential coefficients of an acyclic function of the coordinates.

For, if we draw any closed circuit and suppose it occupied by a conducting wire,

$$\int\left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right)ds,$$

taken round the circuit, must be zero, as otherwise a permanent electric current would be set up in the wire without any expenditure of energy, which is impossible.

Hence $\int(Xdx + Ydy + Zdz)$ between two points must be independent of the path, and therefore

$$Xdx + Ydy + Zdz = -dV,$$

and V must be acyclic.

234. **Distribution of Electricity on Conductors.**—If a conductor be in electric equilibrium, there can be no electromotive force acting in it, and therefore the potential is constant throughout. In the surrounding dielectric,

$$4\pi f = -k\frac{dV}{dx}, \ \&c.,$$

and, accordingly, from (5) we have $\nabla^2 V = 0$. The potential V is therefore determined in the same manner as on the hypothesis of action at a distance.

The charge on a conductor is $\int (lf + mg + nh)\, dS$; and, by (1), Art. 225, this is equal to

$$-\frac{k}{4\pi}\int \frac{dV}{d\nu} dS,$$

where D and ν are both drawn into the dielectric surrounding the conductor.

Hence, if the total charge be given, so also is $\displaystyle\int \frac{dV}{d\nu} dS$.

235. **Conditions at Boundary between two Dielectrics.**—If two dielectrics, whose specific inductive capacities are k_1 and k_2, be in contact, at the boundary between them in passing from one to the other, V is continuous, as otherwise the electromotive intensity perpendicular to the boundary would be infinite.

Again, the normal component of the displacement must be the same in one medium as in the other. To prove this, let us suppose two small tubes of induction resting on the same element of the boundary surface and drawn one in each medium. Let D_1 and D_2 denote the displacements, Σ_1 and Σ_2 orthogonal sections of the tubes drawn close to the boundary surface S, and ψ_1 and ψ_2 the angles between the lines of displacement and the normal to S. Then, by Art. 228, we have $D_1\Sigma_1 = D_2\Sigma_2$; but $\Sigma_1 = dS\cos\psi_1$, $\Sigma_2 = dS\cos\psi_2$, and therefore $D_1\cos\psi_1 = D_2\cos\psi_2$.

The conditions stated above give the equations

$$V_1 = V_2, \quad k_1\frac{dV_1}{d\nu_1} + k_2\frac{dV_2}{d\nu_2} = 0. \qquad (10)$$

If the positive direction be that of the normal drawn into the medium whose inductive capacity is k_1, the second equation above may be written $k_1 N_1 = k_2 N_2$; whence

$$N_1 - N_2 = \frac{k_2 - k_1}{k_1} N_2. \qquad (11)$$

If we suppose k_2 to be greater than k_1, we see that a dielectric of greater inductive capacity, relatively to one of less, behaves like a conductor on which there is a charge of density $\frac{k_2 - k_1}{4\pi k_1} N_2$. In the case of a conductor, we must suppose k_2 infinite, then from (10) $N_2 = 0$. In what precedes, N_1 and N_2 denote the components of electromotive intensity normal to the boundary.

236. **Attraction on Dielectric in Field of Force.**—If a body composed of dielectric material be placed in a medium of different specific inductive capacity, the body in general behaves like a conductor in tending to move.

To see the reason of this we must remember that, in general, if a conductor or a dielectric of different inductive capacity be introduced into a medium occupying a field of force, the total energy of the field is altered; and, unless the field be uniform, the alteration is different according to the part of the field into which the conductor or dielectric is introduced.

If a small change in the position of the conductor diminishes the total energy of the field, the conductor will have a tendency to move in the direction, producing a change of position whereby the total energy of the field is diminished.

The same thing holds good in the case of a body composed of dielectric material differing in inductive capacity from the medium by which the field of force is occupied.

237. **Crystalline Dielectric.**—In a crystalline, or anisotropic, dielectric different directions differ in their electric properties, and the electromotive force is not necessarily co-directional with the displacement.

In this case, the components of the one are linear functions of those of the other, so that we have

$$\left.\begin{aligned} 4\pi f &= k_{11}X + k_{12}Y + k_{13}Z, \\ 4\pi g &= k_{21}X + k_{22}Y + k_{23}Z, \\ 4\pi h &= k_{31}X + k_{32}Y + k_{33}Z. \end{aligned}\right\} \qquad (12)$$

If there be a function U of the components of force, representing per unit of volume the energy due to the displacement, we have

$$\delta U = X\delta f + Y\delta g + Z\delta h. \qquad (13)$$

Substituting from (12) for δf, &c., in (13), and arranging, we get

$$4\pi\delta U = (k_{11}X + k_{21}Y + k_{31}Z)\delta X + (k_{12}X + k_{22}Y + k_{32}Z)\delta Y$$
$$+ (k_{13}X + k_{23}Y + k_{33}Z)\delta Z;$$

but
$$\delta U = \frac{dU}{dX}\delta X + \frac{dU}{dy}\delta Y + \frac{dU}{dz}\delta Z,$$

and therefore

$$k_{21} = k_{12}, \quad k_{32} = k_{23}, \quad k_{31} = k_{13};$$

and
$$8\pi U = k_{11}X^2 + k_{22}Y^2 + k_{33}Z^2 + 2k_{12}XY$$
$$+ 2k_{23}YZ + 2k_{13}XZ; \qquad (14)$$

also,
$$f = \frac{dU}{dX}, \quad g = \frac{dU}{dY}, \quad h = \frac{dU}{dZ}. \qquad (15)$$

By transformation of coordinates, $8\pi U$ can be reduced to the form

$$k_1X^2 + k_2Y^2 + k_3Z^2.$$

When U is reduced to this form, the coordinate axes are the principal axes of electric displacement, and k_1, k_2, k_3 denote the principal inductive capacities of the dielectric.

For the components of electric displacement we have, then, the equations

$$4\pi f = k_1 X, \quad 4\pi g = k_2 Y, \quad 4\pi h = k_3 Z. \qquad (16)$$

If we take any point P of the dielectric as origin and draw the ellipsoid whose equation referred to the principal axes is

$$k_1 x^2 + k_2 y^2 + k_3 z^2 = \varepsilon^2,$$

it is plain that if we draw a line through P in the direction of the electromotive intensity, and draw a tangent plane to the ellipsoid at the point in which it is met by this line, the perpendicular on this tangent plane is in the direction of the electric displacement.

238. **Differential Equation for Potential in Crystalline Medium.**—If we express the principal components of displacement in terms of the electromotive intensity by (16), equation (5) becomes

$$k_1 \frac{dX}{dx} + k_2 \frac{dY}{dy} + k_3 \frac{dZ}{dz} = 0,$$

and therefore

$$k_1 \frac{d^2 V}{dx^2} + k_2 \frac{d^2 V}{dy^2} + k_3 \frac{d^2 V}{dz^2} = 0. \qquad (17)$$

239. **Distribution of Electricity on Conductors.**—As there is no electromotive intensity in the substance of a conductor in electric equilibrium, V must be constant at the surface.

In the dielectric outside the surface, V must satisfy the equation

$$k_1 \frac{d^2 V}{dx^2} + k_2 \frac{d^2 V}{dy^2} + k_3 \frac{d^2 V}{dz^2} = 0.$$

Also, the product of V and its differential coefficient integrated over a sphere of infinite radius must vanish.

This appears from the consideration that $lf + mg + nh$ integrated over a sphere of infinite radius is finite. Hence,

if R denote the radius of the sphere, f must be of the order $\frac{1}{R^2}$; but f, g, h are of the same order as the differential coefficients of V. Accordingly, V is of the order $\frac{1}{R}$, and $V\frac{dV}{dx}$, &c., are of the order $\frac{1}{R^3}$.

Finally, if the charge on each conductor be assigned,

$$\int\left(k_1 l\frac{dV}{dx} + k_2 m\frac{dV}{dy} + k_3 n\frac{dV}{dz}\right)dS$$

is given for each conductor.

There is only one function V which satisfies these conditions.

If there were two, let ϕ be the difference between them. Take the expression

$$k_1\left(\frac{d\phi}{dx}\right)^2 + k_2\left(\frac{d\phi}{dy}\right)^2 + k_3\left(\frac{d\phi}{dz}\right)^2,$$

and integrate it by parts throughout the whole of space—the first term with respect to x, the second with repect to y, and the third with respect to z; then we get

$$-\iiint\left\{k_1\left(\frac{d\phi}{dx}\right)^2 + k_2\left(\frac{d\phi}{dy}\right)^2 + k_3\left(\frac{d\phi}{dz}\right)^2\right\}d\mathfrak{S}$$

$$= \Sigma\iint\phi\left(k_1 l\frac{d\phi}{dx} + k_2 m\frac{d\phi}{dy} + k_3 n\frac{d\phi}{dz}\right)dS$$

$$+\iiint\phi\left(k_1\frac{d^2\phi}{dx^2} + k_2\frac{d^2\phi}{dy^2} + k_3\frac{d^2\phi}{dz^2}\right)d\mathfrak{S}.$$

From what precedes, it is plain that each term on the right-hand side of this equation is zero; and, as k_1, k_2, and k_3 are always positive, we have

$$\frac{d\phi}{dx} = \frac{d\phi}{dy} = \frac{d\phi}{dz} = 0,$$

and therefore ϕ is constant, and consequently zero for the whole of space.

240. **Energy expressed as Surface Integral.**—If V denote the potential, and W the energy due to the electric displacement, by an integration similar to that employed in the last Article, by (14), Art. 236, we obtain

$$8\pi W = \int \left\{ k_1 \left(\frac{dV}{dx}\right)^2 + k_2 \left(\frac{dV}{dy}\right)^2 + k_3 \left(\frac{dV}{dz}\right)^2 \right\} d\mathfrak{S}$$

$$= - \Sigma \int V \left(l k_1 \frac{dV}{dx} + m k_2 \frac{dV}{dy} + n k_3 \frac{dV}{dz} \right) dS$$

$$- \int V \left\{ k_1 \frac{d^2 V}{dx^2} + k_2 \frac{d^2 V}{dy^2} + k_3 \frac{d^2 V}{dz^2} \right\} d\mathfrak{S}.$$

Hence, by (16) and (17), we have

$$2W = \Sigma \int V (lf + mg + nh) \, dS. \qquad (18)$$

Since V is constant at the surface of each conductor, and since $\int (lf + mg + nh) \, dS$ denotes the total charge on the conductor, equation (18) may be written

$$2W = \Sigma e V. \qquad (19)$$

241. **Energy due to Electrified Particle in Electric Field.**—Let us suppose the field to be due to a single conductor, whose surface may be denoted by S_1, on which there is a charge e_1. Let a conductor, whose surface may be denoted by S_2, on which there is a charge e_2, be introduced into the field. Let V denote the potential at any part of the field before the introduction of S_2, and $V + v$ the potential afterwards; also, let W and $W + w$ denote the total energy of the field before and after the introduction of S_2. Then we have

$$2W = V_1 e_1, \quad 2(W + w) = (V_1 + v_1) e_1 + (V_2 + v_2) e_2.$$

If we now suppose S_2 and e_2 to be infinitely small, so also is v, and the term $v_2 e_2$ is of the second order, and therefore negligible. Hence we have

$$2w = V_2 e_2 + v_1 e_1.$$

On the hypothesis that S_2 and e_2 are infinitely small, we have $V_2e_2 = v_1e_1$; for, if we integrate the expression

$$\int\left(k_1\frac{dV}{dx}\frac{dv}{dx} + k_2\frac{dV}{dy}\frac{dv}{dy} + k_3\frac{dV}{dz}\frac{dv}{dz}\right)d\mathfrak{S},$$

since V and v each satisfy equation (17), we get

$$\int V\left(lk_1\frac{dv}{dx} + mk_2\frac{dv}{dy} + nk_3\frac{dv}{dz}\right)\left(dS_1 + dS_2\right)$$

$$= \int v\left(lk_1\frac{dV}{dx} + mk_2\frac{dV}{dy} + nk_3\frac{dV}{dz}\right)\left(dS_1 + dS_2\right). \quad (20)$$

At the surface S_1 the potential V is constant, and

$$\int\left(lk_1\frac{dv}{dx} + mk_2\frac{dv}{dy} + nk_3\frac{dv}{dz}\right)dS_1$$

is zero, since the introduction of S_2 does not alter the total charge on S_1. Again, before the introduction of the conductor S_2 the total charge on the space surface S_2 was zero, and therefore

$$\int\left(lk_1\frac{dv}{dx} + mk_2\frac{dv}{dy} + nk_3\frac{dv}{dz}\right)dS_2 = -4\pi e_2.$$

Hence, as S_2 is infinitely small,

$$\int V\left(lk_1\frac{dv}{dx} + mk_2\frac{dv}{dy} + nk_3\frac{dv}{dz}\right)dS_2$$

cannot differ from $-4\pi V_2e_2$ by more than an infinitely small quantity of the second order. Accordingly the left-hand side of (20) is equal to $-4\pi V_2e_2$.

Again, as $V+v$ and V are each constant at S_1, so also is v, and therefore

$$\int v\left(lk_1\frac{dV}{dx} + mk_2\frac{dV}{dy} + nk_3\frac{dV}{dz}\right)dS_1 = -4\pi v_1e_1.$$

Also, at the surface S_2 we have

$$\int\left(lk_1\frac{dV}{dx}+mk_2\frac{dV}{dy}+nk_3\frac{dV}{dz}\right)dS_2=0\,;$$

and, since $V+v$ is constant, and V can vary only by an infinitely small quantity, the variation of v must be infinitely small, and

$$\int v\left(lk_1\frac{dV}{dx}+mk_2\frac{dV}{dy}+nk_3\frac{dV}{dz}\right)dS_2,$$

taken over the infinitely small surface S_2, cannot differ from zero by more than an infinitely small quantity of the second order. Accordingly the right-hand side of (20) is equal to $-4\pi v_1 e_1$.

We have, then, $V_2 e_2 = v_1 e_1$; and therefore $2w = 2V_2 e_2$. Hence, by bringing an electric particle e_2 to a point where the potential of the field is V_2, the energy produced is $V_2 e_2$.

It is obvious that the result arrived at above can be extended to an electric field due to any number of conductors, so that in general Ve denotes the energy produced by bringing a small body having a charge of electricity e to a point where the potential is V.

If, instead of supposing a small charged conductor introduced into the field, we suppose the charge on one of the conductors S_1, already in the field, increased by the amount δe_1, we can show in a manner similar to that employed above that δW, the increase of energy, is given by the equation

$$\delta W = V_1\delta e_1.$$

In fact,

$$2W = V_1e_1 + V_2e_2 + V_3e_3 + \&c.,$$

and

$$2\delta W = V_1\delta e_1 + e_1\delta V_1 + e_2\delta V_2 + \&c.\,;$$

but

$$V_1\delta e_1 = e_1\delta V_1 + e_2\delta V_2 + e_3\delta V_3 + \&c.,$$

as may be shown in the following manner.

Let the original potential at any point of the field be denoted by V, and the increase of potential due to the introduction of δe_1 by v; then, by an integration similar to that already employed, we have

$$\int V\left(k_1 l \frac{dv}{dx} + k_2 m \frac{dv}{dy} + k_3 n \frac{dv}{dz}\right)\left(dS_1 + dS_2 + \&c.\right)$$

$$= \int v\left(k_1 l \frac{dV}{dx} + k_2 m \frac{dV}{dy} + k_3 n \frac{dV}{dz}\right)\left(dS_1 + dS_2 + \&c.\right).$$

At each of the conductors V is constant, and also $V + v$, and therefore v.

Again, at each conductor, except the first,

$$\int\left(k_1 l \frac{dv}{dx} + k_2 m \frac{dv}{dy} + k_3 n \frac{dv}{dz}\right) dS$$

is zero, and at the first this integral is $-4\pi\delta e_1$.

Also, at each conductor,

$$\int\left(k_1 l \frac{dV}{dx} + k_2 m \frac{dV}{dy} + k_3 n \frac{dV}{dz}\right) dS = -4\pi e;$$

hence the equation above becomes

$$4\pi V_1 \delta e_1 = 4\pi\,(e_1 v_1 + e_2 v_2 + \&c.),$$

that is,

$$V_1 \delta e_1 = e_1 \delta V_1 + e_2 \delta V_2 + e_3 \delta V_3 + \&c. \qquad (21)$$

Hence we obtain

$$\delta W = V_1 \delta e_1, \qquad (22)$$

and therefore we conclude that under any circumstances the energy produced by bringing a small quantity of electricity e to a point where the potential is V is denoted by Ve.

242. **System of Charged Conductors.**—It is now easy to see that, if there be a system of charged conductors in a crystalline dielectric, the equations which hold good

between the charges and potentials are of the same form as those belonging to an isotropic medium, that is,

$$\begin{aligned} V_1 &= p_{11}e_1 + p_{12}e_2 + p_{13}e_3 + \&c., \\ V_2 &= p_{12}e_1 + p_{22}e_2 + p_{23}e_3 + \&c., \\ V_3 &= p_{13}e_1 + p_{23}e_2 + p_{33}e_3 + \&c., \\ &\&c., \qquad \&c. \end{aligned}$$

In fact, every step in the process by which these equations are proved in Art. 128 holds good here.

For, from equation (16), it appears that if each component of displacement be altered in the same ratio, so also are the differential coefficients of the potential and the total displacement. Accordingly, if the mode of distribution of the displacement be assigned, the potential at any one point varies as the displacement to which it is due. Also two systems of displacement which are each in equilibrium may be superposed without disturbing the equilibrium.

243. **Force on Electric Particle in Electric Field.**—We have seen that the energy due to an electric particle e in an electric field is Ve.

If the particle receive a displacement whose components are δx, δy, and δz, the energy of the field is increased by

$$e\left(\frac{dV}{dx}\,\delta x + \frac{dV}{dy}\,\delta y + \frac{dV}{dz}\,\delta z\right).$$

This is the work done against the forces of the field which must therefore be

$$-e\,\frac{dV}{dx}, \quad -e\,\frac{dV}{dy}, \quad \text{and} \quad -e\,\frac{dV}{dz}.$$

Accordingly, the force acting on an electric particle per unit of mass is the same as the electromotive force of the field.

244. **Potential due to Spherical Conductor.**—If a charged spherical conductor whose radius is a be alone in the field, the potential V is constant at the surface of the sphere, and in the space outside satisfies equation (17). If we assume

$$\sqrt{k_1}\,\xi = x, \quad \sqrt{k_2}\,\eta = y, \quad \text{and} \quad \sqrt{k_3}\,\zeta = z,$$

(17) becomes
$$\frac{d^2V}{d\xi^2} + \frac{d^2V}{d\eta^2} + \frac{d^2V}{d\zeta^2} = 0\,; \qquad (23)$$

and, at the surface of the sphere,

$$k_1\xi^2 + k_2\eta^2 + k_3\zeta^2 = a^2. \tag{24}$$

We have therefore to find a function of ξ, η, and ζ which satisfies (23), and which is constant when ξ, η, ζ satisfy (24). This is the same problem as to find the potential of an ellipsoidal charged conductor. Hence the form of V is given by Ex. 3, Art. 75.

If $k_1 > k_2 > k_3$, we have

$$a^2 = \frac{a^2}{k_3}, \quad b^2 = \frac{a^2}{k_2}, \quad c^2 = \frac{a^2}{k_1};$$

$$h^2 = \frac{k_2 - k_3}{k_2 k_3}\, a^2, \;\; k^2 = \frac{k_1 - k_3}{k_1 k_3}\, a^2, \;\; V = C\int_\lambda^\infty \frac{d\lambda}{\sqrt{\{(\lambda^2 - h^2)(\lambda^2 - k^2)\}}}, \tag{25}$$

where λ is the greatest root of the equation

$$\frac{\zeta^2}{\lambda^2} + \frac{\eta^2}{\lambda^2 - h^2} + \frac{\xi^2}{\lambda^2 - k^2} = 1. \tag{26}$$

The constant C is determined from the equation

$$\int\left(lk_1\frac{dV}{dx} + mk_2\frac{dV}{dy} + nk_3\frac{dV}{dz}\right)dS = -4\pi e,$$

where e denotes the charge on the surface S of the conductor.

245. **Force due to Spherical Conductor.**—Differentiating (26) we obtain

$$\frac{2\xi d\xi}{\lambda^2 - k^2} = 2\lambda d\lambda\left\{\frac{\xi^2}{(\lambda^2 - k^2)^2} + \frac{\eta^2}{(\lambda^2 - h^2)^2} + \frac{\zeta^2}{\lambda^4}\right\}.$$

The quantity inside the bracket on the right-hand side may be denoted by $\frac{1}{p^2}$, and we get

$$\frac{d\lambda}{d\xi} = \frac{p^2\xi}{\lambda(\lambda^2 - k^2)}.$$

In like manner we have

$$\frac{d\lambda}{d\eta} = \frac{p^2\eta}{\lambda(\lambda^2 - h^2)}, \quad \frac{d\lambda}{d\zeta} = \frac{p^2\zeta}{\lambda^3}$$

Again,

$$\frac{d\xi}{dx} = \frac{1}{\sqrt{k_1}}, \quad \frac{d\eta}{dx} = 0, \quad \frac{d\zeta}{dx} = 0, \text{ \&c.};$$

accordingly, if X, Y, Z denote the components of force, we have

$$X = -\frac{dV}{dx} = \frac{C}{\sqrt{k_1}} \frac{p^2}{\lambda} \frac{\xi}{\lambda^2 - k^2} \frac{1}{\sqrt{\{(\lambda^2 - h^2)(\lambda^2 - k^2)\}}},$$

$$Y = -\frac{dV}{dy} = \frac{C}{\sqrt{k_2}} \frac{p^2}{\lambda} \frac{\eta}{\lambda^2 - h^2} \frac{1}{\sqrt{\{(\lambda^2 - h^2)(\lambda^2 - k^2)\}}},$$

$$Z = -\frac{dV}{dz} = \frac{C}{\sqrt{k_3}} \frac{p^2}{\lambda} \frac{\zeta}{\lambda^2} \frac{1}{\sqrt{\{(\lambda^2 - h^2)(\lambda^2 - k^2)\}}}.$$

246. **Force due to Spherical Particle.**—In the case of a particle, a becomes infinitely small, and so also do k and h; then

$$\lambda^2 = \lambda^2 - h^2 = \lambda^2 - k^2,$$

and

$$p^2 = \lambda^2 = \xi^2 + \eta^2 + \zeta^2.$$

Accordingly, we have

$$X = \frac{C}{\lambda^2} \frac{\xi}{\sqrt{k_1}\lambda}, \quad Y = \frac{C}{\lambda^2} \frac{\eta}{\sqrt{k_2}\lambda}, \quad Z = \frac{C}{\lambda^2} \frac{\zeta}{\sqrt{k_3}\lambda}.$$

If we substitute for ξ, η, and ζ in terms of x, y, and z, we obtain

$$X = \frac{Cx}{k_1\lambda^3}, \quad Y = \frac{Cy}{k_2\lambda^3}, \quad Z = \frac{Cz}{k_3\lambda^3}, \qquad (27)$$

where

$$\lambda^2 = \frac{x^2}{k_1} + \frac{y^2}{k_2} + \frac{z^2}{k_3}.$$

Hence we conclude that the force exercised by a spherical particle at a point P is not in the direction of the line joining P to the centre of the particle, and does not vary inversely as the square of the distance of P from the centre of the sphere.

From (16) and (27) we have

$$f = \frac{C\mathrm{r}}{4\pi\lambda^3}\,\frac{x}{\mathrm{r}}, \quad g = \frac{C\mathrm{r}}{4\pi\lambda^3}\,\frac{y}{\mathrm{r}}, \quad h = \frac{C\mathrm{r}}{4\pi\lambda^3}\,\frac{z}{\mathrm{r}}, \qquad (28)$$

where r denotes the distance of P from the origin.

Hence, the direction at any point of the displacement due to a spherical particle passes through the centre of the particle, but the magnitude of the displacement does not vary inversely as the square of the distance.

CHAPTER XII.

ELECTROMAGNETIC THEORY OF LIGHT.

247. **Introductory.**—The electromagnetic theory of light cannot be considered part of the theory of Attraction; but it is so intimately connected with the properties of dielectrics, and with those of electricity and magnetism which have been explained in the foregoing chapters, that some account of Maxwell's great investigations does not seem out of place here.

248. **Energy of Current in Magnetic Field.**—From the identity of the action of an electric current with that of a magnetic shell, in Art. 217 it was concluded that the potential energy W of a current in an independent magnetic field is given by the equation

$$W = -i\int(la + mb + nc)\,dS. \tag{1}$$

If i assume the infinitely small value δi, equation (1) becomes

$$\delta W = -\delta i\int(la + mb + nc)\,dS. \tag{2}$$

This equation holds good whether the field be independent or not, as a change in the integral due to an infinitely small value of i must be infinitely small, and when multiplied by δi becomes evanescent. We cannot, however, regard the energy due to the presence of an electric current as potential, because the current is not a permanent natural agent whose action varies merely with its position. The current may cease, and, if so, its energy disappears.

We must therefore consider the energy due to an electric current as kinetic.

In a dynamical system, if work be done against the natural motion of the system, the energy, if potential, is increased, but the energy, if kinetic, is diminished.

Hence we conclude that, if the energy of an electric current be kinetic, the expression for the variation of energy due to a variation of i must have the opposite sign from that which it would have if the energy were potential. Therefore, if T denote the kinetic energy due to the presence of an electric current in a magnetic field, we have

$$\delta T = \delta i \int (la + mb + nc)\, dS. \tag{3}$$

Since $\delta T = \dfrac{dT}{di}\delta i$, we obtain

$$\frac{dT}{di} = \int (la + mb + nc)\, dS. \tag{4}$$

249. **Energy and Electromotive Force.**—The connexion between variation of energy and force is given by Lagrange's Equations, Dynamics, Art. 207.

In the present case of the dynamical system consisting of electric currents in a magnetic field, the position of the system is specified by the geometrical coordinates of the various magnets and electric circuits, and in the case of each current, by the distances along the circuit which the electric molecules have travelled at any time since a definite epoch. If s denote the distance along the circuit which a molecule of electricity has travelled, its velocity is $\dfrac{ds}{dt}$. Again, if ρ denote the density of the electricity, and σ the area of a section of the circuit, the quantity of electricity which passes the section in the unit of time is $\rho\sigma\dfrac{ds}{dt}$, but this, Art. 212, is i the strength of the current. Hence,

$$\int_{t_0}^{t} i\, dt = \int_{t_0}^{t} \rho\sigma \frac{ds}{dt}\, dt = \int_{s_0}^{s} \rho\sigma\, ds = \psi(s) - \psi(s_0),$$

since ρ is constant, and σ a function of s. Hence, if $\int_{t_0}^{t} i\, dt$

and s_0 be assigned so also is s. Accordingly, instead of specifying the position of a molecule of electricity by s we may do so by the coordinate η, where $\eta = \int_{t_0}^{t} i\,dt$. Again, since i is uniform throughout the circuit, η is the same for all the molecules of electricity.

If now X', Y', Z' denote the components of the total electric force at any point of the circuit, Lagrange's equation of motion, corresponding to the coordinate η, is

$$\frac{d}{dt}\frac{dT}{d\dot{\eta}} - \frac{dT}{d\eta} = \Sigma\left(X'\frac{dx}{d\eta} + Y'\frac{dy}{d\eta} + Z'\frac{dz}{d\eta}\right),$$

and if X, Y, Z denote the components of electromotive intensity, the corresponding forces X'', Y'', Z'' are given by the equations

$$X'' = \rho\sigma X ds, \quad Y'' = \rho\sigma Y ds, \quad Z'' = \rho\sigma Z ds,$$

also
$$d\eta = \dot{\eta} dt = \rho\sigma ds.$$

Hence

$$\Sigma\left(X''\frac{dx}{d\eta} + Y''\frac{dy}{d\eta} + Z''\frac{dz}{d\eta}\right)$$

$$= \Sigma\, ds\left\{X\rho\sigma\frac{dx}{\rho\sigma ds} + Y\rho\sigma\frac{dy}{\rho\sigma ds} + Z\rho\sigma\frac{dz}{\rho\sigma ds}\right\}$$

$$= \Sigma\left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) = \int (X dx + Y dy + Z dz) = E.$$

From Art. 212 it appears that when a current is passing the electromotive force is opposed by the resistance of the circuit, so that the generalized component of force tending to increase η is not E but $E - Ri$.

Again, the kinetic energy T does not depend on η but on $\dot{\eta}$ or i. Hence $\dfrac{dT}{d\eta}$ is zero always, and Lagrange's equation of motion corresponding to the generalized coordinate η becomes

$$\frac{d}{dt}\frac{dT}{di} = E - Ri. \qquad (5)$$

This equation may be written

$$E - \frac{d}{dt}\frac{dT}{di} = Ri. \qquad (6)$$

If T remain unchanged, (6) becomes Ohm's equation (2), Art. 212.

If T vary in consequence of a change in the electromagnetic field, the electromotive force keeping up the current is diminished by $\frac{d}{dt}\frac{dT}{di}$. If this be negative, the electromotive force is increased.

This property of currents is abundantly confirmed by experiment. It is indeed on this property that almost all the modern applications of electricity depend. It was originally discovered by observation; but its exact mathematical expression as given above is due to Maxwell.

A simple case of this phenomenon is exhibited if two currents which repel one another be made to approach. An additional electromotive force is then developed in each circuit tending to increase the current.

This still holds good if E be originally zero in one circuit. A current is then produced tending to oppose the motion. Such currents are called 'induction currents.' It is on their existence that the whole theory of light as an electromagnetic phenomenon depends.

The general principle exemplified in the production of induction currents may be expressed by the statement

In any circuit contained in an electromagnetic field every variation in the strength of the field produces an electromotive force which tends to diminish the variation.

250. **Maxwell's Theory of Light.**—Maxwell supposes the entire universe to be filled with a dielectric called the luminiferous ether.

If there be a variable electric displacement in any part of this dielectric, the variation of the displacement constitutes an electric current which produces an electromagnetic field. The variation of the current produces an electromotive force in all

the surrounding circuits. These electromotive forces produce currents which again give rise to other electromotive forces and currents, and so the original variable displacement is propagated through space.

In the case of light, the original displacement is vibratory; that is, it begins in a certain direction, increases in that direction up to a certain amount, and afterwards takes place in the opposite direction till it reaches the same amount as before, only in the opposite direction, when it is again reversed; and this process is repeated so long as the light remains steady.

The displacement is therefore quantitatively the same as the distance moved through by a vibrating particle, and may be represented by an expression of the form

$$a \sin \frac{2\pi}{\tau} t.$$

The whole phenomenon may therefore be termed an electric vibration; and, when propagated through space, may be called an electric wave.

From the results already arrived at, the laws which govern this propagation may be deduced, as will be shown in the following Articles.

In the study of an electric vibration we have to do with five vector quantities: the displacement, the electromotive intensity, the current intensity, the magnetic force, and the magnetic induction.

Let

f, g, h denote the components of electric displacement;
X, Y, Z those of electromotive intensity;
u, v, w those of current intensity;
α, β, γ those of magnetic force;
a, b, c those of magnetic induction.

We seek to determine differential equations for the components of one of the vectors which will enable us to arrive at the laws of its propagation.

251. **Magnetic Induction and Electromotive Intensity.**—We have seen, Art. 249, that for any circuit s, if X, Y, Z denote the components of electromotive intensity *due to current induction*,

$$-\frac{d}{dt}\frac{dT}{di} = \int\left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right)ds. \qquad (7)$$

If we imagine a surface-sheet S filling up the circuit s, by Stokes's theorem, Art. 192, the right-hand side of (7) is equal to

$$\int\left\{l\left(\frac{dZ}{dy} - \frac{dY}{dz}\right) + \&c.\right\} dS;$$

and, by (4),

$$\frac{dT}{di} = \int (la + mb + nc)\, dS.$$

Hence,

$$\frac{d}{dt}\int (la + mb + nc)\, dS$$

$$= \int\left\{l\left(\frac{dY}{dz} - \frac{dZ}{dy}\right) + m\left(\frac{dZ}{dx} - \frac{dX}{dz}\right) + n\left(\frac{dX}{dy} - \frac{dY}{ds}\right)\right\} dS.$$

In the case of an electric disturbance in a continuous medium, this equation holds good for every circuit which can be drawn; and therefore we have

$$\frac{da}{dt} = \frac{dY}{dz} - \frac{dZ}{dy}, \quad \frac{db}{dt} = \frac{dZ}{dx} - \frac{dX}{az}, \quad \frac{dc}{dt} = \frac{dX}{dy} - \frac{dY}{dx}. \qquad (8)$$

252. **Current Intensity and Magnetic Force.**—If we suppose a surface-sheet S drawn in the dielectric, the total current passing across it is denoted by

$$\int (lu + mv + nw)\, dS.$$

The line integral of the magnetic force, taken round a circuit s, bounding the surface S, is due altogether to the current passing across S, since for magnetic forces due to

currents not embraced by s this line integral is zero. Hence, by Art. 215, we have

$$4\pi \int (lu + mv + nw)\, dS = \int \left(a \frac{dx}{ds} + \beta \frac{dy}{ds} + \gamma \frac{dz}{ds} \right) ds$$

$$= \int \left\{ l \left(\frac{d\gamma}{dy} - \frac{d\beta}{dz} \right) + m \left(\frac{da}{dz} - \frac{d\gamma}{dx} \right) + n \left(\frac{d\beta}{dx} - \frac{da}{dy} \right) \right\} dS;$$

and, since this equation holds good for every circuit and corresponding surface which can be drawn in the medium, we have

$$4\pi u = \frac{d\gamma}{dy} - \frac{d\beta}{dz}, \quad 4\pi v = \frac{da}{dz} - \frac{d\gamma}{dx}, \quad 4\pi w = \frac{d\beta}{dx} - \frac{da}{dy}. \qquad (9)$$

253. **Relation between Magnetic Force and Induction.**—We have seen, Art. 201, that in a body magnetically isotropic, in which there is no permanent magnetism, the components of magnetic induction are in a constant ratio to those of magnetic force, so that

$$a = \varpi a, \quad b = \varpi\beta, \quad c = \varpi\gamma. \qquad (10)$$

In what follows, we shall always suppose the medium to be magnetically isotropic.

254. **Equations of the Electromagnetic Field and of Propagation of Disturbance.**—In the general case of a dielectric electrically crystalline, collecting the results given by (16), Art. 237, by Art. 227, and by (9), (10), and (8) of the present Chapter, we have the following group of equations holding good in the electromagnetic field:—

$$4\pi f = K_1 X, \quad 4\pi g = K_2 Y, \quad 4\pi h = K_3 Z. \qquad (11)$$

$$\frac{df}{dt} = u, \quad \frac{dg}{dt} = v, \quad \frac{dh}{dt} = w. \qquad (12)$$

$$4\pi u = \frac{d\gamma}{dy} - \frac{d\beta}{dz}, \quad 4\pi v = \frac{da}{dz} - \frac{d\gamma}{dx}, \quad 4\pi w = \frac{d\beta}{dx} - \frac{da}{dy}. \qquad (13)$$

$$a = \varpi a, \quad b = \varpi\beta, \quad c = \varpi\gamma. \qquad (14)$$

$$\frac{da}{dt} = \frac{dY}{dz} - \frac{dZ}{dy}, \quad \frac{db}{dt} = \frac{dZ}{dx} - \frac{dX}{dz}, \quad \frac{dc}{dt} = \frac{dX}{dy} - \frac{dY}{dx}. \qquad (15)$$

By differentiation from (12) and by (13), &c., we have

$$4\pi\frac{d^2f}{dt^2}=4\pi\frac{du}{dt}=\frac{d}{dt}\left(\frac{d\gamma}{dy}-\frac{d\beta}{dz}\right)=\frac{1}{\varpi}\left(\frac{d}{dy}\frac{dc}{dt}-\frac{d}{dz}\frac{db}{dt}\right)$$

$$=\frac{1}{\varpi}\left\{\frac{d}{dy}\left(\frac{dX}{dy}-\frac{dY}{dx}\right)-\frac{d}{dz}\left(\frac{dZ}{dx}-\frac{dX}{dz}\right)\right\},$$

that is,

$$4\pi\varpi\frac{d^2f}{dt^2}=\nabla^2X-\frac{d}{dx}\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)$$

$$=4\pi\left\{\frac{1}{K_1}\nabla^2f-\frac{d}{dx}\left(\frac{1}{K_1}\frac{df}{dx}+\frac{1}{K_2}\frac{dg}{dy}+\frac{1}{K_3}\frac{dh}{dz}\right)\right\}.$$

If we assume

$$A^2=\frac{1}{\varpi K_1},\quad B^2=\frac{1}{\varpi K_2},\quad C^2=\frac{1}{\varpi K_3},$$

we get

$$\frac{d^2f}{dt^2}=A^2\nabla^2f-\frac{d}{dx}\left(A^2\frac{df}{dx}+B^2\frac{dg}{dy}+C^2\frac{dh}{dz}\right). \qquad (16)$$

In the case of an isotropic medium,

$$A^2=B^2=C^2=V^2,$$

and we have

$$\frac{d^2f}{dt^2}=V^2\nabla^2f. \qquad (17)$$

255. Solution of Equation of Propagation.— Equations (16) and (17) are very general in their character; and to obtain a solution suitable for the present investigation we must consider some of the characteristics of a ray of light.

When light emanating from a point passes through a lens whose focus is at the luminous point, a cylindrical beam is obtained whose parallel sections are planes having similar characteristics in reference to the beam. We may assume therefore that one of the vibrations which constitutes the light is propagated so that its direction remains parallel

to a line fixed in space, and that at all points of a section of the beam parallel to a certain definite direction the vibrations are in parallel directions, and in a similar state or phase. Consequently, if D denote one of the displacements whose vibrations constitute the light, the direction of D is constant, and the direction in which D is propagated through space is also constant.

If λ, μ, ν denote the direction-cosines of D, we may therefore assume that λ, μ, ν are constant for all positions of D, and we have

$$f = \lambda D, \quad g = \mu D, \quad h = \nu D.$$

Equation (17) assumes its simplest form when f is a function of one coordinate; and, as a particular case of (17), we may write

$$\frac{d^2 f}{dt^2} = V^2 \frac{d^2 f}{dz^2}. \qquad (18)$$

By Art. 53, the solution of (18) is

$$f = \phi\,(Vt - z).$$

This expression for f indicates a variable quantity whose magnitude at a given point is continually altering and whose every state or phase advances through space in the direction of z with a velocity V.

This is obvious, because

$$\phi\,\{V(t + t') - (z + z')\} = \phi\,(Vt - z), \quad \text{provided} \quad Vt' = z';$$

and, accordingly, the value of f at the point z at the time t is the same as the value of f at the point $z + z'$ at the time $t + t'$.

If ϕ be a periodic function, the disturbance in the medium is called *a wave.*

The distance between two points on the line of propagation at which the disturbance is in the same state is called *the wave-length.*

If τ denote the period of the disturbance, that is, the length of time in which the disturbance at a fixed point P

passes through all its phases and returns to its original state, the wave-length is equal to $V\tau$. For, during the period τ, the original disturbance reaches a point Q whose distance from P is $V\tau$, and the disturbance at P has during the same time returned to its original state. Hence, at the end of the period τ the disturbance at Q is in the same state or phase as that in which it is at P, and consequently PQ is a wave-length.

When a wave is passing through a medium, the locus of the points at which the disturbance is in the same phase is called *the wave-front*.

If the wave-front be a plane parallel to a plane fixed in space, the wave is called *a plane wave*.

In the case of a plane wave, the direction of propagation is the normal to the wave-front, and the direction of vibration is parallel to a line fixed in space.

We can now generalize the solution of (18) so as to satisfy (17), and to represent the propagation of a plane wave of electric displacement through the dielectric.

We may assume

$$f = \lambda D, \quad g = \mu D, \quad h = \nu D, \quad D = \phi\{Vt - (lx + my + nz)\}, \qquad (19)$$

where l, m, n denote the direction-cosines of a line fixed in space.

Then $\quad \nabla^2 D = (l^2 + m^2 + n^2)\phi'' = \phi''$, and

$$\frac{d^2 D}{dt^2} = V^2 \phi'';$$

and, accordingly, (17) and the corresponding equations for g and h are satisfied, also D represents the displacement in a plane wave whose line of propagation is in the direction l, m, n.

256. **Direction of Displacement in Isotropic Medium.**—The expressions for $\frac{df}{dt}$, &c., given by (12) and (13), show that

$$\frac{d}{dt}\left(\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}\right) = 0.$$

If there be an electric displacement in the medium before the disturbance takes place, by (5), Art. 228,

$$\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0. \tag{20}$$

Hence this equation always holds good; but

$$f = \lambda\phi\{Vt - (lx + my + nz)\},$$

with corresponding equations for g and h, and therefore by (20),

$$(\lambda l + \mu m + \nu n)\phi' = 0,$$

and accordingly

$$\lambda l + \mu m + \nu n = 0,$$

and we learn that in a plane wave the disturbance is perpendicular to the wave-normal, and is therefore in the wave-front.

This is often expressed by saying that the disturbance is in the plane of the wave.

257. **Magnetic Force in Isotropic Medium.**—By (14), &c., we have

$$\frac{d\alpha}{dt} = \frac{1}{\varpi}\frac{da}{dt} = \frac{1}{\varpi}\left(\frac{dY}{dz} - \frac{dZ}{dy}\right) = \frac{4\pi}{\varpi K}\left(\frac{dg}{dz} - \frac{dh}{dy}\right)$$
$$= 4\pi V^2(m\nu - n\mu)\phi'.$$

Integrating with respect to t, we obtain

$$\alpha = 4\pi V(m\nu - n\mu)D + \text{constant}.$$

As we are concerned only with the magnetic force due to the disturbance, the constant may be neglected, and we have

$$\left.\begin{aligned} \alpha &= 4\pi VD(m\nu - n\mu), \\ \beta &= 4\pi VD(n\lambda - l\nu), \\ \gamma &= 4\pi VD(l\mu - m\lambda). \end{aligned}\right\} \tag{21}$$

Hence the magnetic force is in the plane of the wave and perpendicular to the displacement, and its magnitude H is given by the equation

$$H = 4\pi VD. \tag{22}$$

258. **Crystalline Medium.**—The solution found, Art. 255, for (17) holds good for (16) with some modifications.

In fact, if we assume equations (19) and substitute in (16) and the two corresponding equations, we get

$$\left.\begin{aligned} V^2\lambda &= A^2\lambda - l(A^2l\lambda + B^2m\mu + C^2n\nu), \\ V^2\mu &= B^2\mu - m(A^2l\lambda + B^2m\mu + C^2n\nu), \\ V^2\nu &= C^2\nu - n(A^2l\lambda + B^2m\mu + C^2n\nu). \end{aligned}\right\}$$

In the solution of (17) V is given, and we find tha are indeterminate, provided they fulfil the condition

$$l\lambda + m\mu + \nu n = 0.$$

In the present case, when l, m, n are given, equations determine V^2 and λ, μ, ν. If we eliminate λ, μ, ν from (23), we get a cubic equation to determine V^2. The absolute term of this equation is

$$\begin{vmatrix} A^2(l^2-1) & B^2lm & C^2ln \\ A^2lm & B^2(m^2-1) & C^2mn \\ A^2ln & B^2mn & C^2(n^2-1) \end{vmatrix}.$$

If we call this determinant Q, we have

$$Q = A^2B^2C^2lmn \begin{vmatrix} \dfrac{l^2-1}{l} & m & n \\ l & \dfrac{m^2-1}{m} & n \\ l & m & \dfrac{n^2-1}{n} \end{vmatrix}$$

$$= A^2B^2C^2 \begin{vmatrix} l^2-1 & m^2 & n^2 \\ l^2 & m^2-1 & n^2 \\ l^2 & m^2 & n^2-1 \end{vmatrix} = A^2B^2C^2 \begin{vmatrix} 0 & m^2 & n^2 \\ 0 & m^2-1 & n^2 \\ 0 & m^2 & n^2-1 \end{vmatrix} = 0.$$

Hence one value of V^2 is zero. The corresponding values of λ, μ, ν are proportional to

$$\frac{l}{A^2},\quad \frac{m}{B^2},\quad \frac{n}{C^2};$$

but they have no physical import, as the displacement to which they belong is not propagated through the dielectric.

For each of the values of V^2 which are not zero there is a corresponding set of values of λ, μ, ν, indicating two possible wit...ections of displacement with a given wave-front.

If we multiply the first of equations (23) by l, the second by (2... ...d the third by n, and add, we get

and acc... $$m\mu + n\nu) = (A^2 l\lambda + B^2 m\mu + C^2 \nu n)(1 - l^2 - m^2 - n^2) = 0.$$

and w... $$l\lambda + m\mu + n\nu = 0, \qquad (24)$$

pend...

wa... we infer that the two directions of displacement corresponding to a given plane wave-front lie in the plane of the wave.

If we multiply the first of equations (23) by λ, the second by μ, the third by ν, and add, we get

$$V^2(\lambda^2 + \mu^2 + \nu^2)$$
$$= A^2\lambda^2 + B^2\mu^2 + C^2\nu^2 - (A^2 l\lambda + B^2 m\mu + C^2 n\nu)(l\lambda + m\mu + n\nu),$$

and therefore, by (24), we have

$$V^2 = A^2\lambda^2 + B^2\mu^2 + C^2\nu^2. \qquad (25)$$

If λ_1, μ_1, ν_1; λ_2, μ_2, ν_2 denote the direction-cosines of the two displacements perpendicular to l, m, n, and V_1 and V_2 the corresponding velocities of propagation, we have

$$(A^2 - V_1^2)\lambda_1 = l(A^2 l\lambda_1 + B^2 m\mu_1 + C^2 n\nu_1),$$

with two corresponding equations.

Multiplying the first by λ_2, the second by μ_2, the third by ν_2 and adding, since $l\lambda_2 + m\mu_2 + n\nu_2 = 0$, we get

$$V_1^2(\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2) = A^2\lambda_1\lambda_2 + B^2\mu_1\mu_2 + C^2\nu_1\nu_2.$$

In like manner, we have

$$V_2^2(\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2) = A^2\lambda_1\lambda_2 + B^2\mu_1\mu_2 + C^2\nu_1\nu_2.$$

Consequently, unless $V_1 = V_2$, we obtain

$$\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = 0, \qquad (26)$$

[illegible] (27)

Hence we [illegible] ment belonging to t[illegible] each other in the pl[illegible], and are also conjugate in the ellipsoid whose equation is

$$A^2x^2 + B^2y^2 + C^2z^2 = \text{constant}.$$

Since these two directions are perpendicular and conjugate to each other, they are axes of the section of this ellipsoid made by the wave-plane.

259. **Wave-Surface.**—If a vibratory disturbance emanate from a point O and spread in all directions through a medium surrounding O, the locus of points at which at any time the disturbance is in the same state or phase is called the *wave-surface.*

If the medium surrounding O be isotropic, the disturbance is propagated with equal velocities in all directions, and the wave-surface is a sphere having O as centre.

If the medium be not isotropic, we may suppose a number of small plane waves to start simultaneously from O in all possible directions. Each of these is propagated with a velocity corresponding to the direction of its normal. The envelope at any time of all these plane wave-fronts is the wave-surface corresponding to the medium.

260. **Construction for Wave-Surface of Crystalline Medium.**—When an electric disturbance takes place in a crystalline medium, the equations of Art. 258 enable us to give a construction by which the wave-surface may be obtained.

If we take any period of time t_1, and assume

$$a = At_1, \quad b = Bt_1, \quad c = Ct_1,$$

the ellipsoid, whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

... propagated throug.

For each of the values of V^2 which are nelasticity,' and corresponding set of values of λ, μ, ν, indicati
wit:ections of displacement with a given wave-f.e centre O of
by (2 Ellipsoid, and ... equations (23) the electric dis- ...nent in a plane wave due to the disturbance. Draw a ...nt plane to Fresnel's ellipsoid perpendicular to OP;
and Q be its point of contact, and draw OY perpendicular to plane POQ.

Then OY and OQ are conjugate; and, being also at ar...ght angles to each other, are the axes of the section of Fresnel's ellipsoid.

Let the direction-cosines of OP, OY, and OQ be denoted by λ_1, μ_1, ν_1; λ_2, μ_2, ν_2; λ', μ', ν'; then λ_1, μ_1, ν_1 are proportional to

$$\frac{\lambda'}{a^2}, \quad \frac{\mu'}{b^2}, \quad \text{and} \quad \frac{\nu'}{c^2};$$

and therefore, since $\lambda'\lambda_2 + \mu'\mu_2 + \nu'\nu_2 = 0$, we have

$$a^2\lambda_1\lambda_2 + b^2\mu_1\mu_2 + c^2\nu_1\nu_2 = 0;$$

that is, $$A^2\lambda_1\lambda_2 + B^2\mu_1\mu_2 + C^2\nu_1\nu_2 = 0.$$

Also, $$\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = 0.$$

Hence, by Art. 258, OY must be the second possible direction of displacement in the wave-plane corresponding to OP, and this wave-plane must be POY.

Draw OS in the plane POQ perpendicular and equal to OP; then OS is the wave-normal, and its length is the distance through which the wave-front has advanced in the time t_1. If OT be drawn in the plane QOP perpendicular and equal to OQ, the locus of T for all possible positions of OP

is a surface which touches at T the wave-front perpendicular to OS.

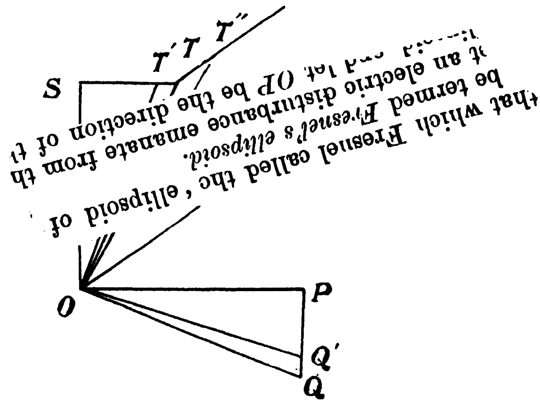

FIG. 1.

To prove this, take on the tangent to Fresnel's ellipsoid, QP, a point Q' infinitely near Q, and in the plane QOP draw OT' perpendicular to OQ'; then $OT' = OQ'$, and if a plane be drawn perpendicular to OT', it passes through OQ', and one axis of the section of Fresnel's ellipsoid made by this plane is infinitely near OQ' and, being an axis, is therefore equal to OQ' and consequently to OT'. Accordingly T' must be a point on the locus surface, and TT' a tangent to this surface.

Again, draw TT'' parallel to OY, and take on it T'' infinitely near T. Then, since TT'' is perpendicular to OT, we have OT'' equal to OT. Again, since OQ is perpendicular to the plane TOT'', the plane perpendicular to OT'' passes through OQ, and the axis of the section of Fresnel's ellipsoid made by this plane, being infinitely near to OQ, is equal to it, and therefore to OT and OT''. Hence T'' is a point on the locus surface, and TT'' a tangent to this surface.

Accordingly, the plane STT'' is a tangent-plane to the locus surface; but this plane is the position of the wave-front at the time t_1. Hence the locus-surface is the envelope of all possible wave-fronts at the time t_1, and is therefore the wave-surface.

261. **Equation of Wave-Surface.**—It is now easy to find, in the manner of MacCullagh, the equation of the wave-surface.

If r denote the length of any radius-vector of Fresnel's ellipsoid, a sphere, having O as centre and r as radius, meets the ellipsoid in the cone whose equation is

$$x^2\left(\frac{1}{r^2} - \frac{1}{a^2}\right) + y^2\left(\frac{1}{r^2} - \frac{1}{b^2}\right) + z^2\left(\frac{1}{r^2} - \frac{1}{c^2}\right) = 0.$$

A tangent plane to this cone meets the ellipsoid in a section in which two consecutive radii vectores are equal to r. Hence the line of contact is an axis of this section, and the extremity of an intercept equal to r on the perpendicular to the tangent-plane to the cone is a point on the wave-surface. If r be regarded as constant, the equation of the cone reciprocal to the cone of intersection of the sphere and ellipsoid is

$$\frac{a^2 r^2}{a^2 - r^2} x^2 + \frac{b^2 r^2}{b^2 - r^2} y^2 + \frac{c^2 r^2}{c^2 - r^2} z^2 = 0. \qquad (28)$$

The coordinates of a point on the wave-surface whose distance from O is r satisfy this equation. Hence, if

$$r^2 = x^2 + y^2 + z^2,$$

equation (28) becomes the equation of the wave-surface.

Rejecting the factor r^2, and getting rid of fractions, we have

$$a^2 x^2 (b^2 - r^2)(c^2 - r^2) + b^2 y^2 (c^2 - r^2)(a^2 - r^2) + c^2 z^2 (a^2 - r^2)(b^2 - r^2) = 0.$$

Arranging in powers of r, and dividing by r^2, we get, finally,

$$(a^2 x^2 + b^2 y^2 + c^2 z^2) r^2 - a^2 (b^2 + c^2) x^2 - b^2 (c^2 + a^2) y^2 - c^2 (a^2 + b^2) z^2 + a^2 b^2 c^2 = 0. \qquad (29)$$

The surface whose equation we have obtained was discovered by Fresnel, and is known as Fresnel's wave-surface.

262. **Magnetic Force.**—From Art. 254, we have

$$\frac{da}{dt} = \frac{1}{\varpi}\frac{da}{dt} = \frac{1}{\varpi}\left(\frac{dY}{dz} - \frac{dZ}{dy}\right) = \frac{4\pi}{\varpi}\left(\frac{1}{K_2}\frac{dg}{dz} - \frac{1}{K_3}\frac{dh}{dy}\right)$$

$$= 4\pi\left(B^2\frac{dg}{dz} - C^2\frac{dh}{dy}\right) = 4\pi(C^2\nu_1 m - B^2\mu_1 n)\phi'.$$

Integrating with respect to t, we get

$$a = \frac{4\pi}{V}(mC^2\nu_1 - nB^2\mu_1)\phi.$$

Hence we have

$$\left.\begin{aligned} a &= \frac{4\pi D}{V}(mC^2\nu_1 - nB^2\mu_1), \\ \beta &= \frac{4\pi D}{V}(nA^2\lambda_1 - lC^2\nu_1), \\ \gamma &= \frac{4\pi D}{V}(lB^2\mu_1 - mA^2\lambda_1). \end{aligned}\right\} \qquad (30)$$

From (30), we see that

$$la + m\beta + n\gamma = 0\,;$$

also,

$$\lambda_1 a + \mu_1\beta + \nu_1\gamma$$

$$= \frac{4\pi D}{V}\{A^2\lambda_1(\mu_1 n - \nu_1 m) + B^2\mu_1(\nu_1 l - \lambda_1 n) + C^2\nu_1(\lambda_1 m - \mu_1 l)\}$$

$$= \frac{4\pi D}{V}(A^2\lambda_1\lambda_2 + B^2\mu_1\mu_2 + C^2\nu_1\nu_2) = 0.$$

Accordingly, the magnetic force is in the wave-plane, and perpendicular to the displacement; that is, its direction coincides with the second possible direction of displacement in the wave-plane.

Hence if H denote the magnetic force, we have

$$H\lambda_2 = \frac{4\pi D}{V}(mC^2\nu_1 - nB^2\mu_1),$$

with two other corresponding equations. Multiplying first by λ_2, second by μ_2, third by ν_2, and adding, we get

$$H = \frac{4\pi D}{V}\{A^2\lambda_1(\mu_2 n - \nu_2 m) + B^2\mu_1(\nu_2 l - \lambda_2 n) + C^2\nu_1(\lambda_2 m - \mu_2 l)\}$$

$$= \frac{4\pi D}{V}(A^2\lambda_1^2 + B^2\mu_1^2 + C^2\nu_1^2) = \frac{4\pi D}{V}V^2 = 4\pi VD. \quad (31)$$

263. **Electromotive Intensity.**—If F denote the resultant electromotive intensity, and θ_1, θ_2, and θ_3 its direction-angles, we have

$$\mathsf{F}\cos\theta_1 = X = \frac{4\pi f}{K_1} = 4\pi\varpi DA^2\lambda_1,$$

with two similar equations; then F is in the direction of OQ, fig. 1; and if χ denote the angle between the displacement and the resultant electromotive intensity, we have

$$\mathsf{F}\cos\chi = \mathsf{F}(\lambda_1\cos\theta_1 + \mu_1\cos\theta_2 + \nu_1\cos\theta_3)$$

$$= 4\pi\varpi D(A^2\lambda_1^2 + B^2\mu_1^2 + C^2\nu_1^2) = 4\pi\varpi V^2 D,$$

and

$$\mathsf{F} = 4\pi\varpi V^2 D\sec\chi. \quad (32)$$

264. **Conditions at a Boundary.**—When a disturbance passes from one medium into another, six conditions must be fulfilled at the boundary; but of these six, only four are independent.

By Art. 228, the normal component of electric displacement must be continuous. Hence, if l, m, n denote the direction-cosines of the normal to the boundary, f, g, h the components of displacement on one side of the boundary-surface, and f', g', h' those on the other, we have

$$l(f - f') + m(g - g') + n(h - h') = 0. \quad (33)$$

Again, the tangential components of electromotive intensity are continuous. In fact, each component of electromotive intensity must be continuous in a direction perpendicular to its own, as otherwise, by (15), there would be an infinite rate of change in the magnetic induction.

Accordingly, if X, Y, Z, and X', Y', Z' denote the components of electromotive intensity at the two sides of the boundary-surface, and λ_1, μ_1, ν_1; λ_2, μ_2, ν_2 the direction-cosines of two mutually perpendicular tangents to the surface, we have

$$\left.\begin{aligned} \lambda_1(X-X') + \mu_1(Y-Y') + \nu_1(Z-Z') = 0, \\ \lambda_2(X-X') + \mu_2(Y-Y') + \nu_2(Z-Z') = 0. \end{aligned}\right\} \tag{34}$$

As the magnetic induction fulfils the solenoidal condition, each of its components must be continuous in a direction coinciding with its own, and therefore the component normal to the surface must be continuous. Hence we have

$$l(a-a') + m(b-b') + n(c-c') = 0. \tag{35}$$

Also, by (13), the components of magnetic force tangential to the surface are continuous, and therefore

$$\left.\begin{aligned} \lambda_1(\alpha-\alpha') + \mu_1(\beta-\beta') + \nu_1(\gamma-\gamma') = 0, \\ \lambda_2(\alpha-\alpha') + \mu_2(\beta-\beta') + \nu_2(\gamma-\gamma') = 0. \end{aligned}\right\} \tag{36}$$

Equation (35) follows from equations (34), as may be shown in the following manner:—

By (15) we have

$$l\left(\frac{da}{dt}-\frac{da'}{dt}\right) + m\left(\frac{db}{dt}-\frac{db'}{dt}\right) + n\left(\frac{dc}{dt}-\frac{dc'}{dt}\right)$$

$$= l\left\{\frac{d}{dz}\left(Y-Y'\right)-\frac{d}{dy}\left(Z-Z'\right)\right\} + m\left\{\frac{d}{dx}\left(Z-Z'\right)-\frac{d}{dz}\left(X-X'\right)\right\}$$

$$+ n\left\{\frac{d}{dy}\left(X-X'\right)-\frac{d}{dx}\left(Y-Y'\right)\right\}.$$

From equations (34) it appears that $X-X'$, $Y-Y'$, and $Z-Z'$ are proportional to $(\mu_1\nu_2-\nu_1\mu_2)$, &c., that is, to l, m, n; or, if $U=0$ be the equation of the boundary-surface to

$$\frac{dU}{dx}, \quad \frac{dU}{dy}, \quad \text{and} \quad \frac{dU}{dz}.$$

Hence, if Λ denote an undetermined function of the coordinates, and Q be put for

$$\frac{1}{\sqrt{\left\{\left(\frac{dU}{dx}\right)^2+\left(\frac{dU}{dy}\right)^2+\left(\frac{dU}{dz}\right)^2\right\}}},$$

we obtain

$$l\frac{d}{dt}\left(a-a'\right)+m\frac{d}{dt}\left(b-b'\right)+n\frac{d}{dt}\left(c-c'\right)$$

$$=l\left(\frac{d}{dz}\Lambda\frac{dU}{dy}-\frac{d}{dy}\Lambda\frac{dU}{dz}\right)+m\left(\frac{d}{dx}\Lambda\frac{dU}{dz}-\frac{d}{dz}\Lambda\frac{dU}{dx}\right)$$
$$+n\left(\frac{d}{dy}\Lambda\frac{dU}{dx}-\frac{d}{dx}\Lambda\frac{dU}{dy}\right)$$

$$=l\left(\frac{dU}{dy}\frac{d\Lambda}{dz}-\frac{d\Lambda}{dy}\frac{dU}{dz}\right)+m\left(\frac{dU}{dz}\frac{d\Lambda}{dx}-\frac{d\Lambda}{dz}\frac{dU}{dx}\right)$$
$$+n\left(\frac{dU}{dx}\frac{d\Lambda}{dy}-\frac{d\Lambda}{dx}\frac{dU}{dy}\right)$$

$$=Q\left\{\frac{dU}{dx}\left(\frac{dU}{dy}\frac{d\Lambda}{dz}-\frac{d\Lambda}{dy}\frac{dU}{dz}\right)+\frac{dU}{dy}\left(\frac{dU}{dz}\frac{d\Lambda}{dx}-\frac{d\Lambda}{dz}\frac{dU}{dx}\right)\right.$$
$$\left.+\frac{dU}{dz}\left(\frac{dU}{dx}\frac{d\Lambda}{dy}-\frac{d\Lambda}{dx}\frac{dU}{dy}\right)\right\}=0.$$

In a similar manner, from equations (12), (13), and (36), we get

$$l\frac{d}{dt}(f-f')+m\frac{d}{dt}(g-g')+n\frac{d}{dt}(h-h')=0.$$

By integration, we obtain

$$l(f-f') + m(g-g') + n(h-h') = \text{constant},$$
$$l(a-a') + m(b-b') + n(c-c') = \text{constant};$$

but as we are here considering only the results of the disturbance, we must suppose f, g, h; f', g', h'; a, &c., to be all initially zero, and therefore we get (33) and (35).

265. **Propagation of Light.**—If we suppose each point of a plane area Σ to be a centre of disturbance, and draw the wave-surfaces of which these points are the centres, and which all correspond to the same period of time t_1, a plane Σ', parallel to Σ, which touches one of these surfaces will touch them all; and if we draw straight lines from the boundary of Σ to the points of contact with Σ' of the surfaces whose centres are on this boundary, the area Σ' enclosed by this cylinder is made up of points at which the disturbances are all in this plane, parallel to one another, and in the same phase. Consequently, Σ' is the wave-plane at the time t_1. Outside the cylinder the plane Σ' does not touch any of the wave-surfaces, and the disturbances due to wave-surfaces corresponding to a period different from t_1 are not in the plane Σ', nor parallel to one another, so that instead of strengthening they interfere with each other. Thus the sensible effect is limited to the area within the cylinder passing through the boundary of Σ; accordingly, the light is propagated in a straight line, and the direction of the cylindrical beam or ray is that of a line drawn from the centre of one of the wave-surfaces to its point of contact with Σ'.

If the medium be isotropic, the wave-surfaces are spheres, and the ray of light is perpendicular to the wave-plane.

If the medium be not isotropic, the ray is in general not perpendicular to the wave-plane.

266. **Reflexion and Refraction.**—When a disturbance advancing through a medium reaches the boundary of another adjoining medium, the continuity of propagation is interrupted. The most general hypothesis we can make is, that disturbances, starting from the boundary, are set up in both media. A small portion of the boundary between the two media may be regarded as a plane area, and we may suppose

a cylindrical ray of disturbance to reach this area. The plane containing the wave-normal of the incident ray and the normal to the boundary is called the *plane of incidence.*

All the plane sections of the cylindrical ray which are parallel to the plane of incidence have a common perpendicular lying in the tangent plane to the boundary.

We shall suppose at first that each medium is isotropic.

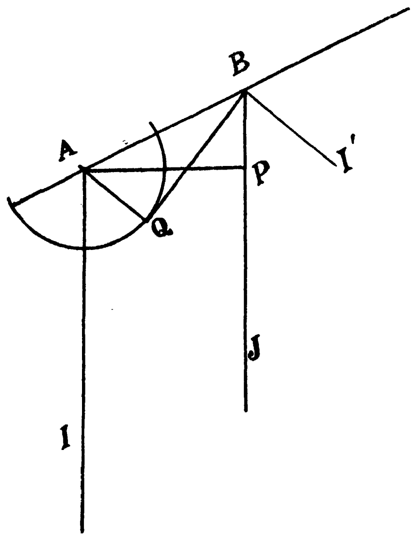

FIG. 2.

Let AB be the line in which the boundary-surface is met by that plane of incidence which contains the longest of the parallel chords of the cylindrical beam of light. Let IA and JB be the lines of intersection of this plane with the cylindrical boundary of the beam.

Draw AP perpendicular to IA. When the disturbance reaches A, wave-surfaces start from A in each medium; and when the disturbance at P reaches B, the wave-surface starting from A is a sphere having A as centre, and a radius equal to PB. There are corresponding wave-surfaces having their centres at all the points of the beam which lie on the plane

boundary of the two media. If we draw through B a perpendicular to the plane of incidence, a plane, through this line, touching the sphere having A as centre, touches all the wave-surfaces, and is therefore the wave-front of the reflected beam. A perpendicular to this plane will be in the direction of the reflected ray.

If BQ be a tangent to the section of the wave-surface starting from A, the reflected ray is in the direction of AQ.

We see, then, from the equality of the triangles AQB and BPA, that the incident and reflected rays make equal angles with the normal to the boundary-surface. We have seen above that the reflected ray lies in the plane of incidence. We have thus the two laws of reflexion in an isotropic medium. The direction of the refracted ray is obtained by a method similar to that employed for the reflected.

Describe, with A as centre, the wave-surface belonging to the second medium and corresponding to the period of time required by the incident ray to travel from P to B. If the velocity of propagation in the second medium is less than in the first, the sphere in the second medium will have a radius AQ' less than PB; and if i and i_1 be the angles which the incident and refracted rays make with the normal to the boundary-surface, we see that the refracted ray is in the plane of incidence, and that

$$\frac{\sin i_1}{\sin i} = \frac{AQ'}{BP} = \frac{V_1}{V},$$

where V and V_1 denote the velocities of propagation in the first and in the second medium.

The ratio $\dfrac{V}{V_1}$ is called 'the index of refraction of the two media'; and if we denote it by μ, we have $\sin i = \mu \sin i_1$.

When the second medium is crystalline, its wave-surface will have two sheets, and two tangent planes can be drawn passing through the perpendicular at B to the plane of incidence. The corresponding directions of displacement are obtained by means of Art. 260, and the lines from A to the points of contact of the tangent planes are the directions of the rays. In a crystalline medium there is thus double refraction, and a single ray of light becomes, in general, two rays.

267. **Common Light and Polarized Light.**—In an isotropic medium the direction of displacement may be any whatever perpendicular to the ray. In the case of common light, the direction of displacement is not fixed, but after a few hundred vibrations passes into another direction in the wave-plane. In the case of light, some billions of vibrations are completed during a second, so that in any appreciable length of time we may consider that there are as many vibrations in any one direction in the wave-plane as in any other. When light is polarized, *all* the vibrations belonging to a given ray are in the same direction.

We have seen that when light passes into a crystalline medium it necessarily becomes polarized. In fact, when the direction of the ray is given, the tangent-plane to the wave-surface at the point where it is met by the ray is the wave-front, and the line in which this plane is met by the plane containing the ray and the wave-normal is the direction of vibration.

268. **Intensity of Light.**—The ultimate measure of the intensity of light is its effect on the eye, but indirectly we can ascertain how it depends on the displacement producing the light and obtain its mathematical expression.

It is ascertained experimentally that if light emanate from a constant source, the intensity of the illumination of a small plane area perpendicular to the direction of the light varies inversely as the square of the distance from the source.

We conclude from this that the intensity of light varies as the energy of the disturbance per unit of volume. In fact, if a disturbance emanates from a source O in an isotropic medium and spreads equally in all directions, the mean total kinetic energy remains constant, and the disturbance at any time occupies the space between two spheres whose radii differ by a wave-length. Since the wave-length is very small, the space occupied by the disturbance is represented by $4\pi r^2\lambda$, where λ denotes the wave-length. Hence if the kinetic energy be denoted by T, the energy per unit of volume is $\dfrac{T}{4\pi r^2\lambda}$; and this varies inversely as the square of the distance from the source.

The simplest form of expression for a periodic disturbance producing a plane wave whose front is perpendicular to the axis of x is $a \cos \phi$, where

$$\phi = \frac{2\pi}{\lambda}(Vt - x),$$

and V denotes the velocity of wave-propagation. The corresponding velocity v of vibration is

$$-\frac{2\pi V}{\lambda} a \sin \phi.$$

Hence the mean value of v^2 is

$$\frac{4\pi^2}{\tau^2} a^2 \frac{1}{2\pi} \int_0^{2\pi} \sin^2\phi \, d\phi \,; \quad \text{that is,} \quad \frac{2\pi^2}{\tau^2} a^2.$$

Accordingly, the density of the medium being constant, the kinetic energy per unit of volume varies as $\frac{\pi^2}{\tau^2} a^2$, or as the square of the amplitude if τ be assigned. If we now suppose a small plane surface to be illuminated by two similar sources of light, the rays from which are approximately perpendicular to the surface, and whose distances from it are equal, the disturbance due to one of these sources may be represented in any direction perpendicular to the ray by $a \cos \phi$, and that due to the other by $a \cos(\phi + \epsilon)$.

The total disturbance is, then, $2a \cos \frac{1}{2}\epsilon \cos(\phi + \frac{1}{2}\epsilon)$. In a short period ϵ passes from O to 2π, and the mean value of the square of the amplitude is

$$\frac{4a^2}{2\pi} \int_0^{2\pi} \cos^2 \tfrac{1}{2}\epsilon \, d\epsilon,$$

which is equal to $2a^2$. Hence, if we suppose that the intensity of light is measured by the energy per unit of volume due to the disturbance, we find that the illumination given by two similar sources of light is double that given by one. Thus the conclusion already arrived at is confirmed.

269. Energy due to Electromagnetic Disturbance.—We have seen (Art. 248) that if T be the kinetic energy of a system of currents in an electromagnetic field,

$$\frac{dT}{di} = \int (la + mb + nc)\, dS.$$

Since T is a homogeneous quadratic function of the strengths of the currents,

$$2T = \Sigma i \frac{dT}{di}, \quad \text{also} \quad a = \frac{dH}{dy} - \frac{dG}{dz}, \text{ \&c.}$$

Substituting in (4), and applying Stokes's theorem, we have

$$i\frac{dT}{di} = i\int\left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right)ds;$$

but

$$i\frac{dx}{ds} = \sigma u,$$

where σ is the orthogonal section of the current, and u the component of its intensity, and $\sigma ds = d\mathfrak{S}$. Hence we get

$$\Sigma i\frac{dT}{di} = \int (Fu + Gv + Hw)\, d\mathfrak{S},$$

where the integral is to be taken throughout the whole of space. Now, by (13),

$$4\pi u = \frac{d\gamma}{dy} - \frac{d\beta}{dz}, \text{ \&c.};$$

whence, substituting, we have

$$8\pi T = \int \left\{F\left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right) + G\left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right) + H\left(\frac{d\beta}{dx} - \frac{d\alpha}{dy}\right)\right\} d\mathfrak{S}$$

$$= \int \{F(m\gamma - n\beta) + G(n\alpha - l\gamma) + H(l\beta - m\alpha)\}\, dS$$

$$+\int \left\{\alpha\left(\frac{dH}{dy} - \frac{dG}{dz}\right) + \beta\left(\frac{dF}{dz} - \frac{dH}{dx}\right) + \gamma\left(\frac{dG}{dx} - \frac{dF}{dy}\right)\right\} d\mathfrak{S},$$

where the volume-integral is to be taken throughout the whole of space, and the surface-integral over both sides of every surface separating two media, and over a sphere whose radius is infinite.

Since α, β, γ are each at infinity of the order $\frac{1}{R^3}$, where R is infinite, the surface-integral at infinity is zero. Again, $m\gamma - n\beta$ is the magnetic force in the plane of yz perpendicular to the normal to the surface S. By (13), such a force, being tangential, is continuous in passing from one side of the surface to the other, and therefore the corresponding surface-integral, when taken over both sides of S, vanishes.

Hence, if we substitute for

$$\frac{dH}{dy} - \frac{dG}{dz}, \text{ \&c.},$$

their equivalents a, b, c, we obtain

$$8\pi T = \int (a\alpha + b\beta + c\gamma)\, d\mathfrak{S} = \varpi \int (\alpha^2 + \beta^2 + \gamma^2)\, d\mathfrak{S}.$$

Substituting for $\alpha^2 + \beta^2 + \gamma^2$ its value from (31), we get

$$8\pi T = 16\pi^2 \int \varpi V^2 D^2 d\mathfrak{S};$$

whence $$T = 2\pi \int \varpi V^2 D^2 d\mathfrak{S}. \tag{37}$$

In addition to the kinetic energy of the electric currents, the disturbance produces potential energy W due to the electric displacement.

If we substitute ϖA^2, ϖB^2, and ϖC^2 for

$$\frac{1}{K_1}, \quad \frac{1}{K_2}, \quad \text{and} \quad \frac{1}{K_3}$$

by equations (14) and (16), Art. 237, we get

$$\begin{aligned} W &= 2\pi \int \varpi (A^2 f^2 + B^2 g^2 + C^2 h^2)\, d\mathfrak{S} \\ &= 2\pi \int \varpi (A^2\lambda^2 + B^2\mu^2 + C^2\nu^2)\, D^2 d\mathfrak{S} = 2\pi \int \varpi V^2 D^2 d\mathfrak{S}. \end{aligned} \tag{38}$$

Hence, if E denote the total energy per unit of volume, we have

$$E = 4\pi\varpi V^2 D^2. \tag{39}$$

270. **Quantities to be lectromagnetic Disturb- and Refraction.**—When ligh that if T be the kinetic into another, there are four quan electromagnetic field by means of the equations holding good at the boundary. These quantities differ according to the nature of the media.

When light passes from one isotropic medium into another, the direction, intensity, and line of displacement of the incident ray being given, the directions of the reflected and refracted rays are known by Art. 266, and we have to determine their intensities and lines of displacement.

When the first medium is isotropic and the second crystalline, the directions and lines of displacement of the two refracted rays are determined by Arts. 266 and 260, and also the direction of the reflected ray. We have, then, to find the intensity and line of displacement of the reflected ray, and the intensities of the two refracted rays.

Similarly, when light passes from a crystalline into an isotropic medium, we have to determine the intensities of the two reflected rays, and the intensity and line of displacement of the refracted.

Lastly, when both media are crystalline, we have to determine the intensities of the two reflected, and of the two refracted rays.

271. **Reflexion and Refraction. Isotropic Media.**—Polarized light passes from one isotropic medium into another: determine the intensities and directions of electric displacement of the reflected and refracted rays.

Let D, D', and D_1 denote the displacements belonging to the incident, reflected, and refracted rays; then we may put

$$D = \mathrm{a} \cos \phi, \quad D' = \mathrm{a}' \cos \phi', \quad D_1 = \mathrm{a}_1 \cos \phi_1,$$

and we may assume that at the surface separating the media

$$\phi = \phi' = \phi_1.$$

Again, if we put

$$I = \sqrt{2\pi\varpi}\, V\mathrm{a}, \quad I' = \sqrt{2\pi\varpi}\, V\mathrm{a}', \quad I_1 = \sqrt{2\pi\varpi_1}\, V_1\mathrm{a}_1,$$

where V and V_1 denote the velocities of propagation of the incident and refracted rays, we have $I^2 = 2\pi\varpi\, V^2\mathrm{a}^2$ = mean value of E, by Art. 269.

where the volume-integral is the intensities of the incident,
whole of space, and the su...
every surface separating t...electrics, ϖ is sensibly the same, so that we may assume $\varpi_1 = \varpi$.

Let the normal to the separating surface drawn into the second medium be the axis of X, and the plane of incidence the plane of XY. Then, by Art. 266, the axis of Z is the line of intersection of the three wave-planes.

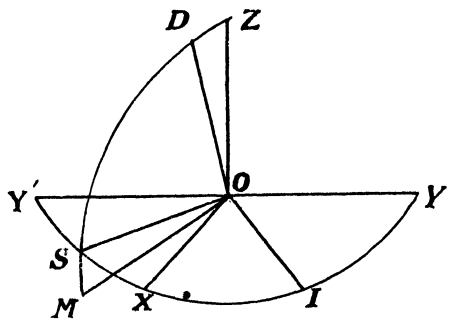

FIG. 3.

Let OI be a wave-normal or ray, and the plane $OZDSM$ the corresponding wave-plane, OD the direction of displacement, and OM, perpendicular to OD, the corresponding direction of magnetic force.

Equations (21) show that the magnetic force is perpendicular to the wave-normal and to the displacement, and is so directed that seen from it the wave-normal must be turned counter-clockwise in order to coincide with the displacement.

Let the displacements make angles θ, θ', and θ_1 with OZ, and let the wave-normals make angles i, i', and i_1 with OX; then, by Art. 266, we have

$$i' = \pi - i, \quad \sin i_1 = \frac{V_1}{V} \sin i,$$

where V and V_1 denote the velocities of wave-propagation in the first and in the second medium.

If X, Y, Z; X', Y', Z'; X_1, Y_1, Z_1 denote the components of electromotive intensity, and α, β, γ; α', β', γ'; $\alpha_1, \beta_1, \gamma_1$ those of magnetic force corresponding to the three rays, by Art. 264 we have

$$Y + Y' = Y_1, \quad Z + Z' = Z_1, \quad \beta + \beta' = \beta_1, \quad \gamma + \gamma' = \gamma_1. \quad (40)$$

By Arts. 257, 254, we have

$$H = 4\pi VD, \quad \mathsf{F} = 4\pi\varpi V^2 D;$$

also, from fig. 3, we see that

$$\left.\begin{aligned} Y &= -\mathsf{F}\sin\theta\cos i, & Y' &= \mathsf{F}'\sin\theta'\cos i, & Y_1 &= -\mathsf{F}_1\sin\theta_1\cos i_1, \\ Z &= \mathsf{F}\cos\theta, & Z' &= \mathsf{F}'\cos\theta', & Z_1 &= \mathsf{F}_1\cos\theta_1. \end{aligned}\right\} \quad (41)$$

$$\left.\begin{aligned} \beta &= -H\cos\theta\cos i, & \beta' &= H'\cos\theta'\cos i, & \beta_1 &= -H_1\cos\theta_1\cos i_1, \\ \gamma &= -H\sin\theta, & \gamma' &= -H'\sin\theta', & \gamma_1 &= -H_1\sin\theta_1. \end{aligned}\right\} \quad (42)$$

As stated above, we may assume $\varpi_1 = \varpi$, and if in equations (40) the members of the first two be divided each by $2\sqrt{2\pi\varpi}$, and the members of the last two be multiplied each by $\dfrac{\sqrt{\varpi}}{2\sqrt{2\pi}}$, these equations by (41) and (42), when $\dfrac{\sin i}{\sin i_1}$ is substituted for $\dfrac{V}{V_1}$, become

$$\left.\begin{aligned} \sin i\cos i\,(I\sin\theta - I'\sin\theta') &= I_1\sin i_1\cos i_1\sin\theta_1, \\ \sin i\,(I\cos\theta + I'\cos\theta') &= I_1\sin i_1\cos\theta_1, \\ \cos i\,(I\cos\theta - I'\cos\theta') &= I_1\cos i_1\cos\theta_1, \\ I\sin\theta + I'\sin\theta' &= I_1\sin\theta_1. \end{aligned}\right\} \quad (43)$$

From the first and last of equations (43) we obtain

$$\left.\begin{aligned} 2I\sin\theta &= \frac{\sin 2i + \sin 2i_1}{\sin 2i} I_1\sin\theta_1, \\ 2I'\sin\theta' &= \frac{\sin 2i - \sin 2i_1}{\sin 2i} I_1\sin\theta_1. \end{aligned}\right\} \quad (44)$$

And from the second and third we have

$$\left.\begin{aligned} 2I\cos\theta &= \frac{\sin(i+i_1)}{\sin i\cos i} I_1\cos\theta_1, \\ 2I'\cos\theta' &= \frac{\sin(i_1-i)}{\sin i\cos i} I_1\cos\theta_1. \end{aligned}\right\} \qquad (45)$$

If the displacement of the incident ray be perpendicular to the plane of incidence, $\theta = 0$, and by (44) we have $I_1 \sin\theta_1 = 0$, whence $I_1 = 0$, or $\theta_1 = 0$; but if we adopt the former alternative, by (45) we have $I\cos\theta = 0$, which is impossible. Hence $\theta_1 = 0$. In like manner, we get $\theta' = 0$; and we learn that in this case all the displacements are perpendicular to the plane of incidence. Again, from (45), we have

$$I_1 = \frac{\sin 2i}{\sin(i+i_1)} I, \quad I' = \frac{\sin(i_1-i)}{\sin(i+i_1)} I. \qquad (46)$$

If the displacement of the incident ray be in the plane of incidence, $\theta = \frac{\pi}{2}$, and from (45) we see that

$$\theta_1 = \frac{\pi}{2}, \quad \theta' = \frac{\pi}{2};$$

as otherwise, by 44, we should have $I = 0$. Hence in this case all the displacements are in the plane of incidence. Also, by (44), we have

$$I_1 = \frac{2\sin 2i}{\sin 2i + \sin 2i_1} I, \quad I' = \frac{\sin 2i - \sin 2i_1}{\sin 2i + \sin 2i_1} I. \qquad (47)$$

In this case, if $\sin 2i_1 = \sin 2i$, we have $I' = 0$; that is, there is no reflected ray. When $2i_1 = \pi - 2i$,

$$i_1 = \frac{\pi}{2} - i, \quad \text{and} \quad \frac{\sin i}{\sin i_1} = \tan i;$$

that is, if μ be the index of refraction, $i = \tan^{-1}\mu$; and we learn that if the tangent of the angle of incidence be equal to the index of refraction, there is no reflected ray when the displacement of the incident ray is in the plane of incidence.

The displacements belonging to common light may be resolved each into two components, one in the plane of incidence, and the other perpendicular to that plane.

The whole of the reflected light is produced by the latter displacements when the angle of incidence is $\tan^{-1}\mu$. This light is therefore polarized, and, if it be made to impinge on a second reflecting surface so that the second plane of incidence is perpendicular to the first, there is no reflected ray when the tangent of the angle of incidence at the second reflecting surface is equal to the corresponding index of refraction.

The discovery of polarized light was partly based on the observation of the phenomenon stated above, and common light, when reflected at the angle of incidence $\tan^{-1}\mu$, was said to be polarized in the plane of incidence.

The plane of polarization as thus specified is perpendicular to the direction of the electric displacement which produces the light.

272. **Reflexion and Refraction. Crystalline Medium.**—Polarized light passes from an isotropic into a crystalline medium: find the intensity and direction of displacement of the reflected ray, and the intensities of the two refracted rays.

Adopting a notation similar to that employed in the last Article, and putting D, D', D_1, and D_2 for the displacements belonging to the four rays, we may assume that at the boundary-surface $\phi = \phi' = \phi_1 = \phi_2$.

As before, take the plane of incidence for the plane of

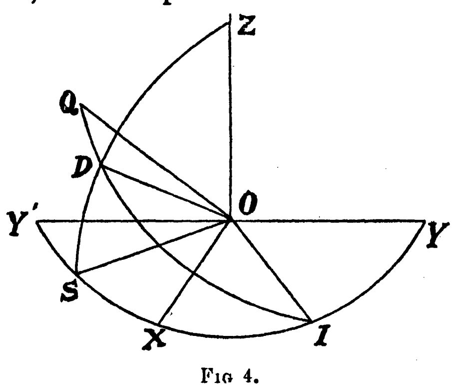

FIG 4.

XY, and the normal to the boundary-surface as the axis of X.

Then all the wave-planes pass through the axis of Z, and all the wave-normals lie in the plane of XY.

Let $ZODS$ be the wave-plane, OI the wave-normal, and OD the direction of displacement corresponding to one of the refracted rays. Then, by Art. 260, the plane IOD contains OQ, the line of direction of the electromotive intensity, and QOD is the angle denoted by χ in Arts. 260 and 263.

Equations (34) and (36) become, in this case,

$$\left.\begin{array}{ll} Y + Y' = Y_1 + Y_2, & Z + Z' = Z_1 + Z_2, \\ \beta + \beta' = \beta_1 + \beta_2, & \gamma + \gamma' = \gamma_1 + \gamma_2. \end{array}\right\} \quad (48)$$

Also, by Art. 263, we have $\mathsf{F}_1 = 4\pi\varpi V^2{}_1 D_1 \sec \chi_1$.

Now $Y_1 = \mathsf{F}_1 \cos Q_1 Y$; but (fig. 4) from the spherical triangle $Q_1 I_1 Y$ we have

$$\cos Q_1 Y = \cos Q_1 I_1 \cos I_1 Y + \sin Q_1 I_1 \sin I_1 Y \cos Q_1 I_1 Y,$$

and $\quad Q_1 I_1 = \dfrac{\pi}{2} + \chi_1, \quad I_1 Y = \dfrac{\pi}{2} - i_1, \quad Q_1 I_1 Y = \pi - DOS = \dfrac{\pi}{2} + \theta_1.$

Hence $\quad \cos Q_1 Y = -\sin \chi_1 \sin i_1 - \cos \chi_1 \cos i_1 \sin \theta_1.$

A similar equation holds good for $Q_2 Y$. Substituting in the value of Y_1, given above, and using I, I', I_1, and I_2 as in Art. 271, we get, instead of Y_1, the expression

$$-I_1 \sin i_1 \sec \chi_1 (\sin \chi_1 \sin i_1 + \cos \chi_1 \cos i_1 \sin \theta_1);$$

that is, $\quad -I_1 (\sin i_1 \cos i_1 \sin \theta_1 + \sin^2 i_1 \tan \chi_1).$

Again, $\quad Z_1 = \mathsf{F}_1 \cos Q_1 Z = \mathsf{F}_1 \cos \chi_1 \cos \theta_1.$

Hence, instead of Z_1, we get $\quad I_1 \sin i_1 \cos \theta_1$.

The expressions to be substituted for the magnetic forces are similar to those made use of in the case of isotropic media. Thus equations (48) become

$$\left.\begin{array}{l} \sin i \cos i (I \sin \theta - I' \sin \theta') \\ \qquad = I_1 (\sin i_1 \cos i_1 \sin \theta_1 + \sin^2 i_1 \tan \chi_1) \\ \qquad\qquad + I_2 (\sin i_2 \cos i_2 \sin \theta_2 + \sin^2 i_2 \tan \chi_2), \\ \sin i (I \cos \theta + I' \cos \theta') = I_1 \sin i_1 \cos \theta_1 + I_2 \sin i_2 \cos \theta_2, \\ \cos i (I \cos \theta - I' \cos \theta') = I_1 \cos i_1 \cos \theta_1 + I_2 \cos i_2 \cos \theta_2, \\ \qquad I \sin \theta + I' \sin \theta' = I_1 \sin \theta_1 + I_2 \sin \theta_2. \end{array}\right\} \quad (49)$$

273. **Uniradial Directions.**—When the angle of incidence is given there are two directions of the displacement belonging to the incident ray for which there is only one refracted ray.

To find one of these directions we may suppose I_2 zero in equations (49), and determine θ in terms of i, i_1, and θ_1.

Making I_2 equal to zero, and eliminating $I' \sin\theta'$ from the first and last of equations (49), we get

$$I \sin 2i \sin\theta = I_1 \{\sin(i + i_1)\cos(i - i_1)\sin\theta_1 + \sin^2 i_1 \tan\chi_1\}. \tag{50}$$

In like manner, from the second and third we obtain

$$I \sin 2i \cos\theta = I_1 \sin(i + i_1)\cos\theta_1.$$

Hence, by division, we get

$$\tan\theta = \cos(i - i_1)\tan\theta_1 + \frac{\sin^2 i_1}{\sin(i + i_1)}\tan\chi_1. \tag{51}$$

The second value of $\tan\theta$ is obtained by putting θ_2 and i_2 for θ_1 and i_1 in (51).

274. **Uniaxal Crystals.**—In the case of what are called uniaxal crystals, Fresnel's ellipsoid is a surface of revolution. If we suppose $c = b$ in the equation of the wave-surface (29), Art. 261, that equation becomes

$$r^2\{a^2x^2 + b^2(y^2 + z^2)\} - a^2b^2(x^2 + y^2 + z^2) - a^2b^2x^2 - b^4(y^2 + z^2) \\ + a^2b^4 = 0;$$

that is, $$(r^2 - b^2)\{a^2x^2 + b^2(y^2 + z^2) - a^2b^2\} = 0. \tag{52}$$

But (52) is the equation of the surface composed of the sphere whose equation is $r^2 = b^2$, and the ellipsoid of revolution whose semi-axis of revolution is b, and whose other semi-axis is a.

If $a > b$, and Fresnel's ellipsoid is prolate, the ellipsoid forming part of the wave-surface is oblate.

These conditions hold good in the case of a crystal of calcium carbonate, commonly called Iceland spar. This crystal is very celebrated in the history of science, as observations of its behaviour led to the discovery of double refraction and of polarized light.

The axis of revolution of Fresnel's ellipsoid is coincident with the line which is called the axis of the crystal. This line is the axis of symmetry, and can be determined from the geometrical form of the crystal.

In the case of an uniaxal crystal, all rays inside the crystal whose directions of electric displacement are perpendicular to the axis are propagated with the same velocity. This appears from (25) by making $C = B$ and $\lambda = 0$; then $\mu^2 + \nu^2 = 1$, and $V^2 = B^2$. Conversely, if

$$A^2 \cos^2 \vartheta + B^2 \sin^2 \vartheta = B^2,$$

we have $(A^2 - B^2) \cos^2 \vartheta = 0$, and therefore $\vartheta = \dfrac{\pi}{2}$. For these rays the wave-surface is a sphere.

Again, if the wave-surface be an ellipsoid of revolution, since the normal to a surface of revolution meets the axis, the ray, the wave-normal, and the axis must be in the same plane; but the plane containing the ray and the wave-normal, by Art. 260, contains the direction of electric displacement. Hence this direction is in the plane containing the ray and the axis.

When light passing through an isotropic medium is refracted at the surface of an uniaxal crystal, one refracted ray is refracted in the same manner as if the crystal were isotropic, since the wave-surface of this ray is a sphere. This ray is called, therefore, the ordinary ray. The other refracted ray, whose wave-surface is an ellipsoid of revolution, is called the extraordinary ray.

Both rays are polarized, and as a result of experiment it is said that the ordinary ray is polarized in the principal plane. By the principal plane is meant the plane passing through the refracted ray and the axis of the crystal. Hence we see again that the direction of electric displacement is perpendicular to the plane of polarization.

275. **Uniaxal Crystal. Reflexion and Refraction.**—In the case of an uniaxal crystal, since $\chi_1 = 0$, equations (49) become

$$\left.\begin{aligned}\sin i\cos i\,(I\sin\theta - I'\sin\theta') &= I_1\sin i_1\cos i_1\sin\theta_1\\ &\quad + I_2(\sin i_2\cos i_2\sin\theta_2 + \sin^2 i_2\tan\chi_2),\\ \sin i(I\cos\theta + I'\cos\theta') &= I_1\sin i_1\cos\theta_1 + I_2\sin i_2\cos\theta_2,\\ \cos i(I\cos\theta - I'\cos\theta') &= I_1\cos i_1\cos\theta_1 + I_2\cos i_2\cos\theta_2,\\ I\sin\theta + I'\sin\theta' &= I_1\sin\theta_1 + I_2\sin\theta_2.\end{aligned}\right\}\quad(53)$$

As an example of the use of these equations, we may suppose light to fall on the surface of an uniaxal crystal cut perpendicular to the axis.

In this case, since the axis of the crystal is the normal to the surface, the plane of incidence contains the axis and the wave-normal of the extraordinary ray, and, consequently, the ray itself. Hence both refracted rays are in the plane of incidence; and, by Art. 274, we have

$$\theta_1 = 0, \quad \theta_2 = \frac{\pi}{2}.$$

Accordingly, equations (53) become

$$\left.\begin{aligned}\sin i\cos i\,(I\sin\theta - I'\sin\theta')&\\ = I_2(\sin i_2\cos i_2 &+ \sin^2 i_2\tan\chi_2),\\ \sin i(I\cos\theta + I'\cos\theta') &= I_1\sin i_1,\\ \cos i(I\cos\theta - I'\cos\theta') &= I_1\cos i_1,\\ I\sin\theta + I'\sin\theta' &= I_2.\end{aligned}\right\}\quad(54)$$

If we now suppose that the incident light is polarized in the plane of incidence, $\theta = 0$; and from the first and last of equations (54) we have

$$I_2(\sin i\cos i + \sin i_2\cos i_2 + \sin^2 i_2\tan\chi_2) = 0.$$

Since the expression by which I_2 is multiplied cannot be zero, we get $I_2 = 0$, and therefore $\sin\theta' = 0$. The second and third of equations (54) become, then,

$$\sin i\,(I + I') = I_1 \sin i_1, \quad \cos i\,(I - I') = I_1 \cos i_1\,;$$

whence we get

$$2I \sin i \cos i = I_1 \sin(i_1 + i), \quad 2I' \sin i \cos i = I_1 \sin(i_1 - i).$$

Finally, we obtain

$$I_1 = I\,\frac{\sin 2i}{\sin(i + i_1)}, \quad I' = I\,\frac{\sin(i_1 - i)}{\sin(i + i_1)}. \qquad (55)$$

Again, if the incident light be polarized in a plane perpendicular to the plane of incidence, $\theta = \frac{\pi}{2}$, and the second and third of equations (54) become

$$I' \sin i \cos\theta' = I_1 \sin i_1, \quad -I' \cos i \cos\theta' = I_1 \cos i_1.$$

Hence we obtain $I_1 \sin(i + i_1) = 0$, and therefore $I_1 = 0$; whence also $\cos\theta' = 0$, and $\theta' = \frac{\pi}{2}$.

From the first and last of equations (54) we have, then,

$$\sin i \cos i\,(I - I') = I_2(\sin i_2 \cos i_2 + \sin^2 i_2 \tan\chi_2), \quad I + I' = I_2.$$

Whence

$$\left.\begin{aligned} I' &= I\,\frac{\sin i \cos i - (\sin i_2 \cos i_2 + \sin^2 i_2 \tan\chi_2)}{\sin i \cos i + \sin i_2 \cos i_2 + \sin^2 i_2 \tan\chi_2}, \\ I_2 &= I\,\frac{2 \sin i \cos i}{\sin i \cos i + \sin i_2 \cos i_2 + \sin^2 i_2 \tan\chi_2}. \end{aligned}\right\} \qquad (56)$$

These expressions can be put into a simpler form.

The angle χ is the angle between the directions of electromotive force and electric displacement, and is measured from the former towards the latter in the same direction as the line of displacement is turned in order to become the wave-normal. This appears from the figures and formulæ of

Arts. 260 and 272. What has been said amounts to this—that in equations (53) χ is to be regarded as positive when the direction of the electromotive force does not lie between those of the displacement and the wave-normal, and consequently the ray does occupy this position. In the present case the axis-minor of the wave-ellipse is the normal to the surface, and the positive angular direction is from it to the refracted wave-normal. The refracted ray lies farther from the axis than the normal, and consequently does not lie between the electric displacement and the wave-normal. Hence in (56) the angle χ_2 is negative.

An expression for $\tan\chi$ can be found by the geometry of the ellipse.

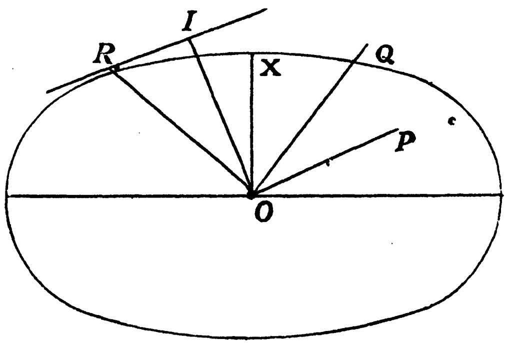

Fig. 5.

In the figure OX represents the axis of the crystal, OQ the line of electromotive force, OP that of electric displacement, OI the extraordinary wave-normal. Then χ is the angle QOP; but $QOP = ROI$, and

$$\tan ROI = \frac{RI}{OI} = \frac{RI \,.\, OI}{OI^2} = \frac{2 \text{ triangle } ROI}{OI^2}.$$

Now, if p_1 and p_2 be the focal perpendiculars on the tangent, and t_1 and t_2 the intercepts on the tangent between their feet and the point of contact, $\dfrac{p_1}{t_1} = \dfrac{p_2}{t_2}$, and therefore

$$(p_1 + p_2)(t_1 - t_2) = (p_1 - p_2)(t_1 + t_2);$$

but $\frac{1}{4}(p_1 + p_2)(t_1 - t_2)$ is double the area of the triangle ROI, and $\frac{1}{4}(p_1 - p_2)(t_1 + t_2)$ is double the area of the right-angled triangle whose sides are $\sqrt{(a^2 - b^2)} \sin i_2$ and $\sqrt{(a^2 - b^2)} \cos i_2$, the angle XOI being i_2. Hence, if OI be denoted by p, we have

$$\tan ROI = \frac{(a^2 - b^2) \sin i_2 \cos i_2}{p^2}.$$

In the present case χ_2 is negative, and we have

$$\sin i_2 \cos i_2 + \sin^2 i_2 \tan \chi_2 = \sin i_2 \cos i_2 \left(1 - \frac{(a^2 - b^2) \sin^2 i_2}{a^2 \sin^2 i_2 + b^2 \cos^2 i_2}\right)$$

$$= \frac{b^2 \sin i_2 \cos i_2}{a^2 \sin^2 i_2 + b^2 \cos^2 i_2} = \frac{B^2 \sin i_2 \cos i_2}{A^2 \sin^2 i_2 + B^2 \cos^2 i_2}.$$

Again, if V denote the velocity of propagation in the external medium,

$$V^2 \sin^2 i_2 = (A^2 \sin^2 i_2 + B^2 \cos^2 i_2) \sin^2 i;$$

whence

$$\sin^2 i_2 = \frac{B^2 \sin^2 i}{V^2 - (A^4 - B^2) \sin^2 i}, \quad \cos^2 i_2 = \frac{V^2 - A^2 \sin^2 i}{V^2 - (A^2 - B^2) \sin^2 i},$$

$$A^2 \sin^2 i_2 + B^2 \cos^2 i_2 = \frac{B^2 V^2}{V^2 - (A^2 - B^2) \sin^2 i}.$$

Hence

$$\frac{B^2 \sin i_2 \cos i_2}{A^2 \sin^2 i_2 + B^2 \cos^2 i_2} = \frac{B \sin i \sqrt{V^2 - A^2 \sin^3 i}}{V^2}, \qquad (57)$$

and

$$\left.\begin{aligned} &\sin i \cos i + \sin i_2 \cos i_2 + \sin^2 i_2 \tan \chi_2 \\ &\qquad = \frac{\sin i \,\{V^2 \cos i + B \sqrt{V^2 - A^2 \sin^2 i}\}}{V^2}, \\ &\sin i \cos i - (\sin i_2 \cos i_2 + \sin^2 i_2 \tan \chi_2) \\ &\qquad = \frac{\sin i \,\{V^2 \cos i - B \sqrt{V^2 - A^2 \sin^2 i}\}}{V^2} \end{aligned}\right\} \qquad (58)$$

Accordingly,

$$\left.\begin{aligned} I' &= I\,\frac{V^2\cos i - B\sqrt{(V^2 - A^2\sin^2 i)}}{V^2\cos i + B\sqrt{(V^2 - A^2\sin^2 i)}},\\ I_2 &= I\,\frac{2V^2\cos i}{V^2\cos i + B\sqrt{(V^2 - A^2\sin^2 i)}}. \end{aligned}\right\} \quad (59)$$

If the value of i be such that $I' = 0$, the reflected ray, when common light falls on the crystal, is polarized in the plane of incidence. This value of i is called the polarizing angle of the crystal when cut perpendicular to its axis.

Making $I' = 0$ in (59), we have

$$V^4(1 - \sin^2 i) = B^2(V^2 - A^2\sin^2 i);$$

whence

$$\sin^2 i = \frac{V^2(V^2 - B^2)}{V^4 - A^2B^2}. \quad (60)$$

276. **Reflexion and Refraction at Interior Surface of Crystal.**—When light passes from the interior of an uniaxal crystal into an isotropic medium, there are, in general, two reflected rays; and when the incident ray is an ordinary ray, we have

$$\left.\begin{aligned} &\sin i_1\cos i_1(I_1\sin\theta_1 - I'_1\sin\theta'_1)\\ &\qquad + (\sin i'_2\cos i'_2\sin\theta'_2 + \sin^2 i'_2\tan\chi'_2)\,I'_2\\ &\qquad\qquad = I_3\sin i_3\cos i_3\sin\theta_3,\\ &\sin i_1(I_1\cos\theta_1 + I'_1\cos\theta'_1) + I'_2\sin i'_2\cos\theta'_2,\\ &\qquad\qquad = I_3\sin i_3\cos\theta_3,\\ &\cos i_1(I_1\cos\theta_1 - I'_1\cos\theta'_1) + I'_2\cos i'_2\cos\theta'_2,\\ &\qquad\qquad = I_3\cos i_3\cos\theta_3,\\ &I_1\sin\theta_1 + I'_1\sin\theta'_1 + I'_2\sin\theta'_2 = I_3\sin\theta_3. \end{aligned}\right\} \quad (61)$$

When the incident ray is an extraordinary ray, the equations at the refracting surface become

$$\left.\begin{aligned}
&(\sin i_2 \cos i_2 \sin \theta_2 + \sin^2 i_2 \tan \chi_2)\, I_2 \\
&\qquad\qquad + I'_1 \sin i'_1 \cos i'_1 \sin \theta'_1 \\
&+ (\sin i'_2 \cos i'_2 \sin \theta'_2 + \sin^2 i'_2 \tan \chi'_2)\, I'_2 \\
&\qquad\qquad = I_3 \sin i_3 \cos i_3 \sin \theta_3, \\
&I_2 \sin i_2 \cos \theta_2 + I'_1 \sin i'_1 \cos \theta'_1 + I'_2 \sin i'_2 \cos \theta'_2 \\
&\qquad\qquad = I_3 \sin i_3 \cos \theta_3, \\
&I_2 \cos i_2 \cos \theta_2 + I'_1 \cos i'_1 \cos \theta'_1 + I'_2 \cos i'_2 \cos \theta'_2 \\
&\qquad\qquad = I_3 \cos i_3 \cos \theta_3, \\
&I_2 \sin \theta_2 + I'_1 \sin \theta'_1 + I'_2 \sin \theta'_2 = I_3 \sin \theta_3.
\end{aligned}\right\} \quad (62)$$

When the crystal is cut perpendicularly to its axis,

$$\theta_1 = \theta'_1 = 0, \quad \theta_2 = \theta'_2 = \frac{\pi}{2}.$$

In this case, the first and last of equations (61) become

$$(\sin i'_2 \cos i'_2 + \sin^2 i'_2 \tan \chi'_2)\, I'_2 = I_3 \sin \theta_3 \sin i_3 \cos i_3,$$
$$I'_2 = I_3 \sin \theta_3.$$

Hence

$$I_3 \sin \theta_3 \{\sin i_3 \cos i_3 - (\sin i'_2 \cos i'_2 + \sin^2 i'_2 \tan \chi'_2)\} = 0\,;$$

but the multiplier of $I_3 \sin \theta_3$ in this equation is not, in general, zero, and therefore we have

$$\sin \theta_3 = 0, \quad I'_2 = 0.$$

Consequently there is no extraordinary reflected ray, and the refracted ray is polarized in the plane of incidence. From the second and third of equations (61) we then obtain

$$I'_1 = I_1 \frac{\sin (i_3 - i_1)}{\sin (i_3 + i_1)}, \quad I_3 = I_1 \frac{\sin 2i_1}{\sin (i_3 + i_1)}. \qquad (63)$$

If the ray incident on the ... be the extraordinary ray, and the crys... ... perpendicularly to its axis, since all the rays ... normals are in the plane of incidence which cuts the wave-ellipsoid in an ellipse whose axis-minor is the axis of the crystal and also the normal to the surface, we have

$$i'_2 = \pi - i_2, \quad \chi'_2 = -\chi_2;$$

whence

$$\sin i'_2 \cos i'_2 + \sin^2 i'_2 \tan \chi'_2 = -(\sin i_2 \cos i_2 + \sin^2 i_2 \tan \chi_2).$$

In this case, the second and third of equations (62) become

$$I'_1 \sin i'_1 = I_3 \sin i_3 \cos \theta_3, \quad I'_1 \cos i'_1 = I_3 \cos i_3 \cos \theta_3,$$

whence $I_3 \cos \theta_3 \sin (i_3 - i'_1) = 0$; but $\sin (i_3 - i'_1)$ cannot be zero, and therefore $\cos \theta_3 = 0$, and $I'_1 = 0$. Consequently, there is no ordinary reflected ray, and the refracted ray is polarized in a plane perpendicular to the plane of incidence.

The first and last of equations (62) now become

$$(\sin i_2 \cos i_2 + \sin^2 i_2 \tan \chi_2)(I_2 - I'_2) = I_3 \sin i_3 \cos i_3,$$
$$I_2 + I'_2 = I_3.$$

Hence

$$I'_2 = I_2 \frac{\sin i_2 \cos i_2 + \sin^2 i_2 \tan \chi_2 - \sin i_3 \cos i_3}{\sin i_2 \cos i_2 + \sin^2 i_2 \tan \chi_2 + \sin i_3 \cos i_3},$$

$$I_3 = I_2 \frac{2(\sin i_2 \cos i_2 + \sin^2 i_2 \tan \chi_2)}{\sin i_2 \cos i_2 + \sin^2 i_2 \tan \chi_2 + \sin i_3 \cos i_3}.$$

By reductions similar to those effected in the case of equations (56) we get

$$\left.\begin{aligned} I'_2 &= I_2 \frac{B\sqrt{(V_3^2 - A^2 \sin^2 i_3)} - V_3^2 \cos i_3}{B\sqrt{(V_3^2 - A^2 \sin^2 i_3)} + V_3^2 \cos i_3}, \\ I_3 &= I_2 \frac{2B\sqrt{(V_3^2 - A^2 \sin^2 i_3)}}{B\sqrt{(V_3^2 - A^2 \sin^2 i_3)} + V_3^2 \cos i_3}. \end{aligned}\right\} \quad (64)$$

Re. ystal, bounded by faces parallel to each
When the inciden ular to the axis, be placed in an isotropic
equations at the ref. of light polarized in a plane perpendicular
... the plane of incidence be transmitted through the crystal, the incident and emergent rays are parallel, and the plane of polarization remains unchanged. Then $i_3 = i$, and in virtue of equations (59) equations (64) become

$$\left.\begin{aligned} I'_2 &= I\,\frac{2V^2\{B\sqrt{(V^2 - A^2\sin^2 i)} - V^2\cos i\}\cos i}{\{B\sqrt{(V^2 - A^2\sin^2 i)} + V^2\cos i\}^2}, \\ I_3 &= I\,\frac{4V^2 B\sqrt{(V^2 - A^2\sin^2 i)}\cos i}{\{B\sqrt{(V^2 - A^2\sin^2 i)} + V^2\cos i\}^2}. \end{aligned}\right\} \quad (65)$$

When the incident ray falls on the first surface of the crystal at the polarizing angle, we have

$$I' = 0, \quad I'_2 = 0, \quad \text{and} \quad I_3 = I_2 = I.$$

In this case, the incident light passes through the crystal unchanged in intensity, direction of electric displacement, and direction of propagation.

277. **Singularities of the Wave-Surface.** — The equation of the wave-surface, Art. 261, may be put in the form

$$(a^2x^2 + b^2y^2 + c^2z^2 - a^2c^2)(x^2 + y^2 + z^2 - b^2) - (a^2 - b^2)(b^2 - c^2)y^2 = 0.$$

From this equation it appears that if the point of intersection of the three surfaces

$$a^2x^2 + b^2y^2 + c^2z^2 - a^2c^2 = 0, \quad x^2 + y^2 + z^2 - b^2 = 0, \quad y = 0$$

be taken as origin, the lowest terms in the equation of the wave-surface are of the second degree, and therefore that the origin is a double point on the wave-surface at which there is a tangent cone of the second degree.

If we seek for the coordinates of the points of intersection of the three surfaces, we have

$$a^2x^2 + c^2z^2 = a^2c^2, \quad x^2 + z^2 = b^2, \quad y = 0;$$

whence we obtain for the coordinates of the point the expressions

$$x^2 = c^2 \frac{a^2 - b^2}{a^2 - c^2}, \quad y^2 = 0, \quad z^2 = a^2 \frac{b^2 - c^2}{a^2 - c^2}. \tag{66}$$

The equation of the circular sections of Fresnel's ellipsoid is

$$x^2 \frac{a^2 - b^2}{a^2} - z^2 \frac{b^2 - c^2}{c^2} = 0;$$

whence, if ϖ_1, ϖ_2, and ϖ_3 denote the direction-cosines of the perpendicular to a plane of circular section, we have

$$\varpi_1^2 = \frac{(a^2 - b^2)c^2}{(a^2 - c^2)b^2}, \quad \varpi_2^2 = 0, \quad \varpi_3^2 = \frac{(b^2 - c^2)a^2}{(a^2 - c^2)b^2}. \tag{67}$$

From (66) and (67) it appears that a singular point on the wave-surface is on a perpendicular to the plane of a circular section of Fresnel's ellipsoid at a distance b from the origin.

The existence of such points follows readily from the mode of generation of the wave-surface described in Art. 260. From thence it appears that the perpendicular to each section of Fresnel's ellipsoid meets the wave-surface in two points whose distances from the centre are equal to the principal semi-axes of the section.

If the section be circular, every axis is a principal axis, and all the corresponding points on the wave-surface coalesce into one.

The perpendiculars on the corresponding tangent-planes of the ellipsoid are, however, not in the same plane; and thus corresponding to the one ray going from the centre to the singular point there are an infinite number of wave-fronts, that is, an infinite number of tangent-planes to the wave-surface meeting at the singular point.

As the wave-normals and velocities of propagation are different for these fronts, when the ray reaches the surface of the crystal it is refracted into an infinite number of rays, forming a cone, and the phenomenon exhibited is termed *conical refraction.*

From the consideration of the ellipsoid reciprocal to Fresnel's ellipsoid, it is easy to see that the wave-surface must possess singularities of another kind in addition to those mentioned above.

From Art. 260, it appears that the perpendicular to each section of the reciprocal ellipsoid is perpendicular to two tangent-planes of the wave-surface, and meets them in points whose distances from the centre are the reciprocals of the semi-axes of the section. If the section be a circular section, every axis is a principal axis, and all the corresponding feet of perpendiculars on tangent-planes to the wave-surface coalesce into one.

The central radii of the reciprocal ellipsoid are co-directional with perpendiculars on tangent-planes of Fresnel's ellipsoid, which are the reciprocals of the radii, so that all the perpendiculars to tangent-planes of Fresnel's ellipsoid which lie in a circular section of the reciprocal ellipsoid are equal to the mean semi-axis of Fresnel's ellipsoid, and correspond to a single tangent-plane to the wave-surface. The corresponding radii of Fresnel's ellipsoid do not, however, lie in the same plane, and are not equal, so that there are an infinite number of rays corresponding to the same wave-front which must therefore touch the wave-surface all along a curve. To find the nature of this curve, we may proceed thus.

Let p denote the length of the central perpendicular on a tangent-plane of Fresnel's ellipsoid, and α, β, γ its direction-angles.

If p lie in the circular section of the reciprocal ellipsoid, we have $p = b$, and therefore

$$a^2 \cos^2\alpha + b^2 \cos^2\beta + c^2\cos^2\gamma = b^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma);$$

that is, $$(a^2 - b^2)\cos^2\alpha - (b^2 - c^2)\cos^2\gamma = 0.$$

Also, $$\cos^2\alpha + \cos^2\gamma = \sin^2\beta;$$

whence $$\cos^2\alpha = \frac{b^2 - c^2}{a^2 - c^2}\sin^2\beta, \quad \cos^2\gamma = \frac{a^2 - b^2}{a^2 - c^2}\sin^2\beta \qquad (68)$$

Let r denote the central radius of Fresnel's ellipsoid to the point of contact of the tangent-plane perpendicular to p, then

$$r^2 = \frac{a^4 \cos^2\alpha + b^4 \cos^2\beta + c^4 \cos^2\gamma}{p^2};$$

and if ρ denotes the distance of this point of contact from the foot of the perpendicular, $\rho^2 = r^2 - p^2$. In the present case, $p = b$, and we have

$$\rho^2 = \frac{a^4 \cos^2\alpha + b^4 \cos^2\beta + c^4 \cos^2\gamma - b^4}{b^2}$$
$$= \frac{(a^4 - b^4)\cos^2\alpha - (b^4 - c^4)\cos^2\gamma}{b^2}.$$

Substituting for $\cos^2\alpha$ and $\cos^2\gamma$ their values from (68), we get

$$\rho^2 = \frac{(a^2 - b^2)(b^2 - c^2)}{b^2}\sin^2\beta.$$

It is plain, from the construction in Art. 260, that ρ is the distance from the foot of the perpendicular on the tangent-plane to the wave-surface to its point of contact, and that this distance is parallel to the corresponding direction of displacement in the wave-plane. In the present case the wave-plane contains the axis of y, and β is the angle which the electric displacement makes with this axis. Hence β is the angle which the line from the foot of the perpendicular to the point of contact of the wave-front with the wave-surface makes with a parallel to the axis of y in the wave-front.

Accordingly,

$$\rho = \frac{\sqrt{(a^2 - b^2)(b^2 - c^2)}}{b}\sin\beta, \qquad (69)$$

is the equation of the curve along which the wave-front touches the wave-surface. This curve is therefore a circle which touches the parallel to the axis of y at the foot of the perpendicular from the centre, and whose diameter is denoted by the expression

$$\frac{\sqrt{(a^2 - b^2)(b^2 - c^2)}}{b}.$$

Corresponding to th... ...ane we have been considering, there are an infinite number of rays which meet the wave-front along its circle of contact with the wave-surface. All these rays have the same wave-normal, and are propagated with the same normal velocity. Hence, when they are refracted at the surface of the crystal, the emergent rays are parallel and form a cylinder. Unless the wave-normal be normal to the surface, the section of this cylinder made by the plane bounding the crystal is an ellipse.

The remarkable phenomena described above were foretold by Hamilton as consequences of properties of Fresnel's wave-surface discovered by him. They were realized experimentally, first by Lloyd, and long afterwards by Fitzgerald.

278. **Total Reflexion.**—When light passes from a denser into a rarer medium, if the angle of incidence exceed $\sin^{-1}\dfrac{1}{\mu}$, where μ denotes the relative index of refraction of the media, there is no refracted ray. In fact, under these circumstances, a refracted wave-plane is impossible, as it would in the case of an isotropic medium, be a tangent-plane to a sphere drawn through a line lying inside the sphere. If both media be isotropic, equations (43) seem impossible to satisfy; for, if we suppose I_1 zero, these equations cannot be satisfied unless we make I and I' each zero.

Mathematically it is possible to give a solution of equations (43), which in its final result is physically satisfactory; but it seems impossible to obtain a sound physical basis for the equations themselves.

The mathematical solution is as follows:—Assume

$$D = \mathrm{a}e^{-\iota\phi}, \quad D' = \mathrm{a}'e^{-\iota\phi'}, \quad D_1 = \mathrm{a}_1 e^{-\iota\phi_1},$$

where $\quad \iota = \sqrt{-1}, \quad$ and $\quad \phi = \dfrac{2\pi}{\lambda}\{Vt - (lx + my + nz)\},$

$$\phi' = \frac{2\pi}{\lambda}\{Vt - (l'x + m'y + n'z)\},$$

$$\phi_1 = \frac{2\pi}{\lambda_1}\{V_1 t - (l_1 x + m_1 y + n_1 z)\};$$

then the differential equations of wave-propagation are satisfied, and D, &c., are periodic.

If we now suppose the incident light polarized in the plane of incidence, since

$$\frac{V_1}{\lambda_1} = \frac{1}{\tau_1}, \quad \frac{V}{\lambda} = \frac{1}{\tau}, \quad \text{and} \quad \tau_1 = \tau,$$

at the origin, where x, y, and z are all zero, we have

$$\phi = \phi' = \phi_1 = \phi_0;$$

and as equations (43) mathematically hold good, we have

$$I' = \frac{\sin i_1 \cos i - \cos i_1 \sin i}{\sin i_1 \cos i + \cos i_1 \sin i} I.$$

But $\qquad \sin i_1 = \mu \sin i, \quad \cos i_1 = \iota\sqrt{(\mu^2 \sin^2 i - 1)},$

and therefore

$$\frac{I'}{I} = \frac{\mu \sin i \cos i - \iota \sin i \sqrt{(\mu^2 \sin^2 i - 1)}}{\mu \sin i \cos i + \iota \sin i \sqrt{(\mu^2 \sin^2 i - 1)}} = \frac{1 - \iota \tan \varepsilon}{1 + \iota \tan \varepsilon}$$

$$= (\cos \varepsilon + \iota \sin \varepsilon)^{-2} = e^{-2\iota\epsilon}, \quad \text{where} \quad \tan \epsilon = \frac{\sqrt{(\mu^2 \sin^2 i - 1)}}{\mu \cos i}.$$

Hence $\qquad D' = \mathrm{a}\, e^{-2\iota\epsilon} e^{-\iota\phi'} = \mathrm{a}\, e^{-\iota(\phi' + 2\epsilon)},$

and, accordingly, the intensity of the reflected light is equal to that of the incident; but its phase is increased by 2ϵ.

Again,

$$\phi_1 = \frac{2\pi}{\lambda_1} \{ V_1 t - (l_1 x + m_1 y + n_1 z) \};$$

and since the axis of x is normal to the surface separating the media, and the axis of z perpendicular to the plane of incidence, we have

$$l_1 = \cos i_1 = \iota\sqrt{(\mu^2 \sin^2 i - 1)}, \quad m_1 = \mu \sin i, \quad n_1 = 0,$$

$$D_1 = \mathrm{a}_1 e^{\iota\phi_1} = \mathrm{a}_1 e^{\frac{2\iota\pi}{\lambda_1}\sqrt{(\mu^2 \sin^2 i - 1)}\iota x}\, e^{-\frac{2\iota\pi}{\lambda_1}(V_1 t - \mu y \sin i)}$$

$$= \mathrm{a}_1 e^{-\frac{2\pi x\sqrt{(\mu^2 \sin^2 i - 1)}}{\lambda_1}}\, e^{-\frac{2\iota\pi}{\lambda_1}(V_1 t - \mu y \sin i)}. \qquad (70)$$

In this expression, the power of e whose index is real is not periodic; and since λ_1 is very small when x is of sensible magnitude, this factor tends to become very small. Hence at any sensible distance from the boundary D_1 is very small, and there is no visible refracted ray.

In other cases of total reflexion a similar mode of treatment may be employed. The results obtained above satisfy the mathematical conditions holding good when the reflexion is not total, and the final result is consistent with the observed phenomena; but the whole investigation can scarcely be regarded as having any physical validity.

279. **Absorption of Light.**—When a medium is not a perfect insulator, an electromotive force produces not only a change of electric displacement but also a conduction-current.

If C be the electric conductibility of the medium, the resistance of an element of unit section parallel to the axis of x is $\frac{dx}{C}$, and the electromotive force for this element is Xdx. Hence, if i_1 denote the intensity of the conduction-current parallel to x, we have $i_1 = CX$.

The total current is made up of the conduction-current and that due to a change of the electric displacement; accordingly, we have

$$u = i_1 + \dot{f} = CX + \dot{f},$$

and as $X = \frac{4\pi}{K} f$, we obtain $u = \dot{f} + \frac{4\pi C}{K} f$.

Substituting for u in terms of the components of magnetic force, and for the latter in terms of those of displacement, by means of equations (13), (15), and (11), we get

$$\ddot{f} + \frac{4\pi C}{K}\dot{f} = \frac{1}{\varpi K}\left\{\nabla^2 f - \frac{d}{dx}\left(\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}\right)\right\}. \quad (71)$$

The last term in this equation is zero; and if we take the normal to the plane of the wave as the axis of z, the displacement f is a function of z only, and (71) becomes

$$\frac{d^2 f}{dt^2} + \frac{4\pi C}{K}\frac{df}{dt} = \frac{1}{\varpi K}\frac{d^2 f}{dz^2}. \quad (72)$$

O

If U denote the velocity of wave-propagation when there is no absorption, we have

$$\frac{1}{\varpi K} = U^2,$$

and putting $\pi\varpi C = k$, we get

$$\frac{4\pi C}{K} = 4kU^2.$$

Thus (72) becomes

$$\frac{d^2f}{dt^2} + 4kU^2\frac{df}{dt} = U^2\frac{d^2f}{dz^2}. \tag{73}$$

To solve this equation, we may assume

$$f = \mathrm{a}\,e^{\iota(nt-mz)},$$

where

$$\iota = \sqrt{-1}, \quad n = \frac{2\pi}{\tau},$$

and m is a quantity to be determined so as to satisfy (73). We have, then,

$$-n^2 + 4kU^2n\iota = -m^2U^2;$$

that is,

$$= \frac{n^2}{U^2} - 4kn\iota. \tag{74}$$

Assume $m = q - \iota p$, then

$$q^2 - p^2 = \frac{n^2}{U^2}, \quad pq = 2kn.$$

Eliminating q, we get

$$p^4 + \frac{n^2}{U^2}p^2 = 4k^2n^2;$$

whence

$$p^2 = \frac{-\frac{n^2}{U^2} \pm \sqrt{\left\{\frac{n^4}{U^4} + 4^2k^2n^2\right\}}}{2}.$$

As p is real, p^2 must be positive, and therefore

$$p^2 = \frac{n^2}{2U^2}\left\{\left(1 + \frac{4^2k^2U^4}{n^2}\right)^{\frac{1}{2}} - 1\right\}.$$

Here k is of the same order as C, which is of the order

$$\frac{K}{\mathcal{T}} \text{ or } \frac{1}{U^2\mathcal{T}}.$$

Hence $\frac{4^2k^2U^4}{n^2}$ is of the form $\nu\frac{\tau^2}{\mathcal{T}^2}$, where ν is a numerical coefficient depending on C and on the units selected, τ the time of vibration in the wave of light, and $\mathcal{T}$ the unit of time. In order that C should have any sensible magnitude, $\mathcal{T}$ must be enormously great compared with τ. Hence $\frac{4^2k^2U^4}{n^2}$ is a small quantity, whose square may be neglected in the expansion of the square root, and we have

$$p^2 = \frac{n^2}{2U^2}\,\frac{4^2k^2U^4}{2n^2} = 4k^2U^2.$$

Substituting $q - \iota p$ for m, we get

$$f = \mathrm{a}\,e^{-pz}\,e^{\iota(nt-qz)}. \tag{75}$$

As the wave is advancing in the direction of z positive, q is positive; and since pq is positive, p must be positive. Hence

$$p = 2kU = 2\pi\varpi CU, \tag{76}$$

also

$$q = \frac{2kn}{p} = \frac{n}{U} = \frac{2\pi}{U\tau},$$

and

$$f = \mathrm{a}\,e^{-pz}\,e^{\frac{2\iota\pi}{U\tau}(Ut-z)}. \tag{77}$$

It follows, from the expression obtained for f, that the velocity of wave-propagation is U, and is therefore unaltered by absorption. In consequence of the factor e^{-pz}, the amplitude of f diminishes as z increases. Since p varies as C, unless C be very small, the amplitude of f diminishes rapidly, and the medium is practically opaque.

280. **Electrostatic and Electromagnetic Measure.**—The reader of the foregoing pages may have been struck by an apparent inconsistency between the present Chapter and Chapter XI.

In Chapter XI. the specific inductive capacity k is of the nature of a numerical quantity. In the present Chapter, the specific inductive capacity K is regarded as the reciprocal of the square of a velocity. The apparent inconsistency results from the fact that in Chapter XI. the various quantities are supposed to be expressed in electrostatic measure, whereas in the present Chapter they are supposed to be expressed in electromagnetic.

We must consider the hypotheses on which the two modes of measurement are based, and how it is that in reference to space, time, and mechanical force, the expression for the same physical quality of a body is in one mode of expression a quantity of a nature different from what it is in the other.

Let e and E denote quantities of electricity expressed in electrostatic and electromagnetic measure, X and $\mathfrak{X}$ the corresponding electromotive intensities, and f and f the displacements. Let L denote a linear magnitude, and T a portion of time, and let us use the symbol = to mean that two expressions denote quantities of the same nature.

Electrostatic measure is based on the assumption that the product of two quantities of electricity divided by the square of a line denotes a mechanical force, that is, $\frac{e^2}{L^2}$ = mechanical force.

Electromagnetic measure is based on the assumption that the product of the strengths of two magnetic poles divided by the square of a line denotes a mechanical force, that is, if m denote the strength of a magnetic pole,

$$\frac{m^2}{L^2} = \text{mechanical force} = \frac{e^2}{L^2}; \quad \text{whence} \quad e = m.$$

Again, i denoting the strength of a current, $E = Ti$; but $i = j$, where j denotes the strength of a magnetic shell, and jL^2 = magnetic moment = mL; whence

$$E = T\frac{m}{L} = T\frac{e}{L}.$$

The electromotive intensity multiplied by a quantity of electricity denotes in either system of measurement a mechanical force; accordingly, $eX = E\mathfrak{X}$; but

$$E = \frac{T}{L}e, \quad \text{and therefore} \quad \mathfrak{X} = \frac{L}{T}X.$$

Again, $\mathrm{f}L^2 = E$, and $fL^2 = e$; whence

$$\mathrm{f} = \frac{T}{L}f, \quad \text{also} \quad f = \frac{e}{L^2} = X,$$

and therefore

$$\mathrm{f} = \frac{T}{L}X = \frac{T^2}{L^2}\mathfrak{X} = \frac{1}{V^2}\mathfrak{X},$$

where V denotes a velocity. Thus k is a numerical quantity, but K the reciprocal of the square of a velocity.

The magnitude of the unit of electricity differs very much in the two systems of measurement.

In the electromagnetic system, two units at the unit of distance apart act on each other with the unit force.

In the electromagnetic system, two magnet-poles of unit strength, at the unit distance apart, act on each other with the unit force.

A circular current of unit strength acts on a unit magnet-pole at its centre with a force which is 2π times the unit of force, provided the radius of the circle be of unit length.

The quantity of electricity which passes through a section of this circuit in the unit of time is the unit quantity of electricity expressed in electromagnetic measure.

The quantity of electricity contained in the electromagnetic unit is n times the quantity contained in the electrostatic.

If, then, E and e denote the same absolute quantity of electricity expressed, one in electromagnetic, the other in electrostatic units, and if L and T denote the units of length and of time, we have

$$E = \frac{1}{n}\frac{T}{L}e;$$

but $E\mathfrak{X} = eX$, where $\mathfrak{X}$ and X denote the electromotive force corresponding to the quantity of electricity denoted by E and e; whence

$$\mathfrak{X} = \frac{nL}{T}X.$$

Then,

$$E = \frac{T}{nL}e = \frac{T}{nL}L^2kX = L^2k\frac{T^2}{n^2L^2}\mathfrak{X} = L^2K\mathfrak{X};$$

whence

$$\frac{1}{K} = n^2\frac{L^2}{T^2}\frac{1}{k}.$$

When the second and centimetre are taken as the units of time and length,

$$n = 3 \times 10^{10} \quad \text{approximately}.$$

NOTE ON THOMSON AND DIRICHLET'S THEOREM,

ARTICLE 70.

When the number of given surfaces is reduced to one, this theorem is proved by Gauss in the following manner:—

(1) On a given surface S a homogeneous distribution of a given quantity of mass is always possible, such that $\int V\sigma dS$ is a minimum. For this distribution, V is constant for all occupied parts of the surface, and there is no part unoccupied.

If r denote the longest distance between any two points of S, and M the total mass, it is obvious that at any point of S the potential cannot be less than $\frac{M}{r}$, since the distribution is homogeneous, that is, composed of mass having everywhere the same algebraical sign. Hence $\int V\sigma dS$ cannot be less than $\frac{M^2}{r}$.

Consequently $\int V\sigma dS$ cannot be diminished without limit, and there must be a distribution such that $\int V\sigma dS$ cannot be made less. In this distribution V must be constant. For, if for an occupied portion Σ_1 of the surface, V_1 be everywhere greater than A, and for another equal portion Σ_2 of the surface V_2 be everywhere less than A, at each point of Σ_1 let $\delta\sigma = -\nu$, and at each point of Σ_2 let $\delta\sigma = +\nu$, then the total mass remains unaltered, and

$$\begin{aligned}\delta \int V\sigma ds = \int V\delta\sigma dS + \int \delta V\sigma dS &= 2\int V\delta\sigma dS \\ &= -2\nu \int (V_1 - V_2)\, dS,\end{aligned}$$

since δV is the distribution resulting from $\delta\sigma$, and therefore, by Art. 51, we have

$$\int \delta V\sigma dS = \int V\delta\sigma dS.$$

Accordingly, $\int V\sigma dS$ has received a variation which is essentially negative, and consequently cannot be a minimum for the distribution σ. Hence, when the integral is a minimum, V is constant for the occupied part of the surface. If there were a part unoccupied by Art. 66, V would be less for this part than for the occupied part, and hence as before the integral could be made less. Accordingly, in the distribution for which $\int V\sigma dS$ is least, there is no part of the surface unoccupied.

(2) If U be a given function of the coordinates, a homogeneous distribution of given mass over S is possible, such that $\int (V-2U)\sigma dS$ is a minimum. For this distribution $V-U$ is constant for all occupied portions of the surface.

If U' be the largest value of U on S, it is clear that $\int (V-2U)\sigma dS$ cannot be less than

$$\left(\frac{M}{r}-2U'\right)M,$$

and therefore that there must be a distribution such that $\int (V-2U)\sigma dS$ cannot be made less. For this distribution $V-U$ is constant at all occupied parts of the surface.

Let $\quad W = \int (V-2U)\sigma dS, \quad$ then

$$\delta W = \int \delta V\sigma dS + \int (V-2U)\,\delta\sigma dS = 2\int (V-U)\,\delta\sigma dS.$$

If $V-U$ be greater than A at every point of an occupied portion of surface Σ_1, and less than A at every point of an equal portion Σ_2 of surface, as in (1), δW can be made negative, and therefore W cannot be the least possible.

In this case, if part of the surface S be unoccupied, $V-U$ may be greater on this part than it is on the occupied part, and therefore in this case we cannot show that the whole surface must be occupied.

(3) Suppose now three distributions of mass on S.

1. A distribution whose surface-density is σ_0 and potential V_0, such that $\int V\sigma dS$ is the least possible, the total mass being M.

2. A distribution whose surface-density is σ_1 and potential V_1, such that $\int (V - \frac{1}{2}\varepsilon U)\, \sigma dS$ is the least possible, the total mass being M, and ε being a given constant.

3. A distribution whose density is σ_2, and potential V_2, such that

$$\sigma_2 = \frac{\sigma_1 - \sigma_0}{\varepsilon}, \quad \text{and} \quad V_2 = \frac{V_1 - V_0}{\varepsilon},$$

then the total mass is zero, and

$$V_2 - U = \frac{V_1 - \varepsilon U - V_0}{\varepsilon},$$

but this is constant for all parts of the surface occupied by σ_1.

If ε be diminished without limit, the distribution σ_1 passes into σ_0, and in this case there is no finite portion of the surface S left unoccupied.

Hence, when ε is diminished without limit, $V_2 - U$ is constant for the whole surface S.

Let us now superpose on σ_2 the distribution whose density is $a\sigma_0$, where a is constant. Then

$$V = aV_0 + V_2, \quad \text{and} \quad V - U = aV_0 + V_2 - U.$$

By a proper determination of a the right-hand member of this equation can be made zero at all points of S.

Accordingly for a single surface Thomson and Dirichlet's Theorem is proved.

This theorem in its most general form can readily be deduced from the properties of fluid motion.

Suppose that the given surfaces S_1, S_2, &c., are surrounded by liquid, or incompressible fluid, of unit density, extending to infinity. Apply to the liquid at each surface an impulsive pressure which at each surface is equal to the given value of Thomson's function for that surface. The liquid begins to move irrotationally, and the velocity potential of the motion is the same as the impulsive pressure, and is equal at each surface to the given value of Thomson's function, and satisfies Laplace's equation for the whole of space.

Thus the truth of Thomson and Dirichlet's Theorem is established.

It is easy to show from Green's theorem that, if there be a given quantity of mass on each of a number of surfaces, this mass may be so distributed that the potential shall be constant over each surface.

Let $2W = \Sigma \int V\sigma dS$. Then

$$8\pi W = -\Sigma \int V\left(\frac{dV}{d\nu} + \frac{dV}{d\nu'}\right)dS = \int \left\{\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right\} d\mathfrak{S}.$$

Hence W is essentially positive, and cannot therefore be diminished without limit, and there must be a distribution of mass such that W cannot be made less. For this distribution V must be constant for each surface. For if V be not constant, W may be made to receive a variation which is negative by transferring positive mass from points on the surface where V is greater than A to points where it is less than A. In this case it is not necessary that the part of the surface from which the transfer is made should be occupied. On the other hand, if V be constant for each surface, any change of distribution increases W. For, let v be the change in V, then

$$8\pi \Delta W = -\Sigma \int \left(V\frac{dv}{d\nu} + v\frac{dV}{d\nu}\right)dS - \Sigma \int v\frac{dv}{d\nu}dS$$

$$= -2\Sigma \int V\frac{dv}{d\nu}dS - \Sigma \int v\frac{dv}{d\nu}dS,$$

where the surface-integrals are to be taken over both sides of the surfaces; but as V is constant for each surface, and

$$\int \frac{dv}{d\nu}dS \text{ zero,}$$

since the total mass is constant, we have

$$8\pi \Delta W = -\Sigma \int v\frac{dv}{d\nu}dS = \int \left\{\left(\frac{dv}{dx}\right)^2 + \left(\frac{dv}{dy}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\} d\mathfrak{S},$$

since $\nabla^2 v = 0$ throughout the field. Hence the change in W is essentially positive, and W is least when V is constant over each surface.

The property of the potential made use of in (1) to show that the whole surface must be occupied is perhaps more readily established by the method of Gauss than by that employed in Art. 66.

Gauss's method is as follows :—

If there be no mass outside a surface S on which the potential is everywhere positive, its value at a point O, outside S, is positive, and less than A its greatest value on S.

For if the potential P at O be greater than A, draw lines in all directions from O; they meet the surface S or go to infinity, and the potential on any one of them passes from P to A, or to some value less than A. Hence on every line there is a point at which the value of the potential is B, lying between P and A. All these points form a closed surface at which the potential is constant; and as there is no mass inside it, the potential has the same value throughout the interior of the surface, Art. 61. Hence the value of the potential at O is B, and cannot consequently be P as was supposed. If P were negative, we could show in like manner that the potential at O must lie between P and zero, and could not therefore have the supposed value.

Again, the potential at O cannot be A or zero. For if it has either of these values, describe a sphere round O as centre. At no point of the surface of this sphere can the potential be greater than A or less than zero. Hence its mean value on this surface cannot be A or zero, unless it have this value for the whole surface of the sphere, in which case it would have the same value for the whole of space external to S, which is impossible.

If the potential be everywhere negative on the surface S, its value at a point O outside S is negative, and less in absolute magnitude than its greatest negative value on S. This is proved in a manner precisely similar to that adopted in the case of the positive potential.

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[*The figures refer to the pages.*]

THE END.